# Trade and Labor Market Dynamics

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## 1 Trade and Labor Market Dynamics Summary

Households have Cobb-Douglas preferences

$$C_t^{nj} = \prod_{k=1}^{J} \left(c_t^{nj,k}\right)^{\alpha^k}, \sum_{k=1}^{J} \alpha^k = 1$$

where $c_t^{nj,k}$ is the consumption of sector $k$ goods in market $nj$ at time $t$. The ideal price index is given by

$$P_t^n = \prod_{k=1}^{J} \left(\frac{P_t^{nk}}{\alpha^k}\right)^{\alpha^k}$$

where $P_t^{nk}$ is the price index of goods purchased from sector $k$ for final consumption in region $n$. Households are forward looking and discount the future rate $\beta \geq 0$. Migration decisions are subject to sectoral and spatial mobility costs. Labor relocation costs are given by $\tau^{nj,ik} \geq 0$ depend on the origin $(nj)$ and destination $(ik)$ are time invariant, additive, and measured in terms of utility. Households also have an additive idiosyncratic shocks for each choice, denoted by $\varepsilon_t^{ik}$.

At each period, households observe the conditions in all labor markets and the realizations of their own idiosyncratic shocks. If they begin the period in a labor market, they work and earn the market wage. At the end of the period, households have the option to relocate. Household decision to relocate is governed by

$$v_t^{nj} = U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[v_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\}, \tag{1}$$

$$\text{s.t. } C_t^{nj} = \frac{w_t^{nj}}{P_t^n}$$

where $v_t^{nj}$ is the lifetime utility of a household currently in region $n$ and sector $j$ at time $t$. We assume that idiosyncratic shocks are i.i.d. over time and distributed Type-1 Extreme Value with mean zero and $\nu$ controls the variance of these shocks.

The expected lifetime utility of a representative agent in market $nj$ at time $t$ is thus given by

$$V_t^{nj} \equiv E\left[v_t^{nj}\right]$$

and using the assumption of the distribution of the shocks, we can rewrite the expected lifetime utility as

$$V_t^{nj} = U\left(C_t^{nj}\right) + \nu \log\left(\beta \sum_{ik} \exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}\right) \tag{2}$$

which is the Bellman equation of our dynamic programming (DP) problem. In a mean-field game, this can be thought of us as the backward equation. This process describes how decisions for the future are made.

The transition matrix governing the probabilities that agents relocate from market $nj$ to $ik$ is determined by

$$\mu_t^{nj,ik} = \frac{\exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{1/\nu}} \tag{3}$$

The transition matrix follows a gravity structure and the parameter $\frac{1}{\nu}$ can be interpreted as the elasticity of migration. The final equation that characterizes our DP problem is given by

$$L_{t+1}^{nj} = \sum_{ik} \mu_t^{ik,nj} L_t^{ik}$$

which describes the evolution of labor markets. Within the context of mean-field game, this would be the Kolmogorov forward equation.

# 2 Dynamic Programming I

In this section, we derive the components of the dynamic programming and its representation using "dynamic-hat algebra".

## 2.1 Deriving equation (2)

We first need to take the expectation of our lifetime utility over the idiosyncratic shocks $ik$.

$$\begin{aligned}
v_t^{nj} &= U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[v_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\} \\
E\left[v_t^{nj}\right] &= E\left[U\left(C_t^{nj}\right) + \max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta E\left[v_{t+1}^{ik}\right] - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\}\right] \\
V_t^{nj} &= U\left(C_t^{nj}\right) + E\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\}\right]
\end{aligned}$$

We are concerned with deriving a closed form solution for this term

$$\Phi_t^{nj} = E\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\}\right]$$

When agents decide to relocate from market $nj$ to $ik$, they need to choose a location that maximizes their expected lifetime utility. Suppose the solutions that maximizes our agent's expected lifetime utility is given by

$$\begin{aligned}
\text{argmax}_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\} &= ik \\
\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\} &= \beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}
\end{aligned}$$

This can only be true if for all $mh \neq ik$, $\beta V_{t+1}^{mh} - \tau^{nj,mh} + \nu \varepsilon_t^{mh} \leq \beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}$. Rewriting this expression, we can say

$$\varepsilon_t^{mh} - \varepsilon_t^{ik} \leq \frac{\beta\left(V_{t+1}^{ik} - V_{t+1}^{mh}\right) - \left(\tau^{nj,ik} - \tau^{nj,mh}\right)}{\nu}$$

let $\bar{\varepsilon}_t^{ik,mh} = \frac{\beta\left(V_{t+1}^{ik} - V_{t+1}^{mh}\right) - \left(\tau^{nj,ik} - \tau^{nj,mh}\right)}{\nu}$, which implies

$$\varepsilon_t^{mh} \leq \bar{\varepsilon}_t^{ik,mh} + \varepsilon_t^{ik}$$

Therefore, the probability that the idiosyncratic shock for every $mh \neq ik$ does not maximize expected future utility is therefore given by the joint probability density $f\left(\varepsilon_t^{ik}\right)\prod_{mh\neq ik} F\left(\bar{\varepsilon}_t^{ik,mh}+\varepsilon_t^{ik}\right)$. Taking the expectation over the continuum of idiosyncratic shocks thus takes the form

$$\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\varepsilon_t^{ik}\right) f\left(\varepsilon_t^{ik}\right)\prod_{mh\neq ik} F\left(\bar{\varepsilon}_t^{ik,mh}+\varepsilon_t^{ik}\right) d\varepsilon_t^{ik}$$

This gives the contribution from a single market. Therefore, by the law of total expectation, the average option value from moving from market $nj$ can be found by summing over all households

$$\Phi_t^{nj}=\sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\varepsilon_t^{ik}\right) f\left(\varepsilon_t^{ik}\right)\prod_{mh\neq ik} F\left(\bar{\varepsilon}_t^{ik,mh}+\varepsilon_t^{ik}\right) d\varepsilon_t^{ik}$$

We assume that the idiosyncratic shocks follow a Type-1 Extreme Value distribution. Substituting the PDF and CDF and with some algebra, we get

$$\begin{aligned}
\Phi_t^{nj} &= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\varepsilon_t^{ik}\right) f\left(\varepsilon_t^{ik}\right)\prod_{mh\neq ik} F\left(\bar{\varepsilon}_t^{ik,mh}+\varepsilon_t^{ik}\right) d\varepsilon_t^{ik}\\
&= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\varepsilon_t^{ik}\right) e^{\left(-\varepsilon_t^{ik}-\gamma\right)} e^{\left(-e^{\left(-\varepsilon_t^{ik}-\gamma\right)}\sum_{ik} e^{\left(-\bar{\varepsilon}_t^{ik,mh}\right)}\right)} d\varepsilon_t^{ik}\\
\lambda_t^{ik} &\equiv \log\left(\sum_{mh} e^{\left(-\bar{\varepsilon}_t^{ik,mh}\right)}\right)\\
\zeta_t^{ik} &= \varepsilon_t^{ik}+\gamma\\
\gamma &\equiv \text{Euler's constant}\\
d\zeta_t^{ik} &= d\varepsilon_t^{ik}\\
\Phi_t^{nj} &= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\varepsilon_t^{ik}\right) e^{\left(-\zeta_t^{ik}\right)} e^{\left(-e^{\left(-\zeta_t^{ik}\right)} e^{\lambda_t^{ik}}\right)} d\varepsilon_t^{ik}\\
&= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\left(\zeta_t^{ik}-\gamma\right)\right)\exp\left(-\zeta_t^{ik}-\exp\left(-\left(\zeta_t^{ik}-\lambda_t^{ik}\right)\right)\right) d\zeta_t^{ik}\\
\tilde{y}_t^{ik} &= \zeta_t^{ik}-\lambda_t^{ik}\\
d\tilde{y}_t^{ik} &= d\zeta_t^{ik}\\
\Phi_t^{nj} &= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\left(\tilde{y}_t^{ik}+\lambda_t^{ik}-\gamma\right)\right)\exp\left(-\left(\lambda_t^{ik}+\tilde{y}_t^{ik}\right)-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\zeta_t^{ik}\\
\Phi_t^{nj} &= \sum_{ik}\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\tilde{y}_t^{ik}+\nu\lambda_t^{ik}-\nu\gamma\right)\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\zeta_t^{ik}\\
\Phi_t^{nj} &= \sum_{ik}\left[\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}-\nu\gamma\right)\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\zeta_t^{ik}\right]\\
&+\sum_{ik}\left[\int_{-\infty}^{\infty}\nu\tilde{y}_t^{ik}\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\zeta_t^{ik}\right]\\
&+\sum_{ik}\left[\int_{-\infty}^{\infty}\nu\lambda_t^{ik}\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\zeta_t^{ik}\right]\\
\Phi_t^{nj} &= \sum_{ik}\left[\int_{-\infty}^{\infty}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}-\nu\gamma\right)\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\tilde{y}_t^{ik}\right]\\
&+\sum_{ik}\left[\int_{-\infty}^{\infty}\nu\tilde{y}_t^{ik}\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right) d\tilde{y}_t^{ik}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{ik}\left[\int_{-\infty}^{\infty}\nu\lambda_t^{ik}\exp\left(-\lambda_t^{ik}\right)\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]\\
\Phi_t^{nj}&=\sum_{ik}\left[\exp\left(-\lambda_t^{ik}\right)\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}-\nu\gamma\right)\int_{-\infty}^{\infty}\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]\\
&+\sum_{ik}\left[\nu\exp\left(-\lambda_t^{ik}\right)\int_{-\infty}^{\infty}\tilde{y}_t^{ik}\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]\\
&+\sum_{ik}\left[\nu\lambda_t^{ik}\exp\left(-\lambda_t^{ik}\right)\int_{-\infty}^{\infty}\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]\\
\Phi_t^{nj}&=\sum_{ik}\left[\exp\left(-\lambda_t^{ik}\right)\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}-\nu\gamma\right)\int_{-\infty}^{\infty}\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]\\
&+\sum_{ik}\left[\nu\gamma\exp\left(-\lambda_t^{ik}\right)\right]\\
&+\sum_{ik}\left[\nu\lambda_t^{ik}\exp\left(-\lambda_t^{ik}\right)\int_{-\infty}^{\infty}\exp\left(-\tilde{y}_t^{ik}-\exp\left(-\tilde{y}_t^{ik}\right)\right)d\tilde{y}_t^{ik}\right]
\end{aligned}
$$

$$
\int_{-\infty}^{\infty}\exp\left(-x-\exp\left(-x\right)\right)dx=1
$$

$$
\begin{aligned}
\Phi_t^{nj}&=\sum_{ik}\exp\left(-\lambda_t^{ik}\right)\left[\beta V_{t+1}^{ik}-\tau^{nj,ik}+\nu\lambda_t^{ik}\right]\\
\Phi_t^{nj}&=\sum_{ik}\left[\exp\left(-\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\right)\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)\right]\\
&+\sum_{ik}\left[\exp\left(-\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\right)\nu\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\right]\\
\bar{\varepsilon}_t^{ik,mh}&=\frac{\beta\left(V_{t+1}^{ik}-V_{t+1}^{mh}\right)-\left(\tau^{nj,ik}-\tau^{nj,mh}\right)}{\nu}
\end{aligned}
$$

$$
\Phi_t^{nj}=\sum_{ik}\left[\exp\left(-\log\left(\sum_{mh}\exp\left(-\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)\frac{1}{\nu}\right)\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)\frac{1}{\nu}\right)\right)\right)\left(\begin{array}{c}\beta V_{t+1}^{ik}-\tau^{nj,ik}\\+\nu\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\end{array}\right)\right]
$$

$$
\Phi_t^{nj}=\sum_{ik}\left[\exp\left(-\log\left(\frac{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)\frac{1}{\nu}\right)}{\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)\frac{1}{\nu}\right)}\right)\right)\left(\begin{array}{c}\beta V_{t+1}^{ik}-\tau^{nj,ik}\\+\nu\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\end{array}\right)\right]
$$

$$
\Phi_t^{nj}=\sum_{ik}\left[\frac{\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)^{\frac{1}{\nu}}\right)}{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)^{\frac{1}{\nu}}\right)}\left(\begin{array}{c}\beta V_{t+1}^{ik}-\tau^{nj,ik}\\+\nu\log\left(\sum_{mh}\exp\left(-\bar{\varepsilon}_t^{ik,mh}\right)\right)\end{array}\right)\right]
$$

$$
\Phi_t^{nj}=\sum_{ik}\left[\frac{\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)^{\frac{1}{\nu}}\right)}{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)^{\frac{1}{\nu}}\right)}\left(\begin{array}{c}\beta V_{t+1}^{ik}-\tau^{nj,ik}\\+\nu\log\left(\frac{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)^{\frac{1}{\nu}}\right)}{\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)^{\frac{1}{\nu}}\right)}\right)\end{array}\right)\right]
$$

$$
\Phi_t^{nj}=\sum_{ik}\left[\frac{\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)^{\frac{1}{\nu}}\right)}{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)^{\frac{1}{\nu}}\right)}\left(\begin{array}{c}\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)-\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)\\+\nu\log\left[\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)\frac{1}{\nu}\right)\right]\end{array}\right)\right]
$$

$$
\Phi_t^{nj}=\left[\frac{\sum_{ik}\exp\left(\left(\beta V_{t+1}^{ik}-\tau^{nj,ik}\right)^{\frac{1}{\nu}}\right)}{\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)^{\frac{1}{\nu}}\right)}\left(\nu\log\sum_{mh}\exp\left(\left(\beta V_{t+1}^{mh}-\tau^{nj,mh}\right)\right)^{\frac{1}{\nu}}\right)\right]
$$

$$\Phi_t^{nj} = \nu \log \sum_{mh} \exp\left(\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)\right)^{\frac{1}{\nu}}$$

Substituting this term back into equation (2), we arrive at our final result

$$V_t^{nj} = U\left(C_t^{nj}\right) + E\left[\max_{\{i,k\}_{i=1,k=0}^{N,J}} \left\{\beta V_{t+1}^{ik} - \tau^{nj,ik} + \nu \varepsilon_t^{ik}\right\}\right]$$

## 2.2 Deriving equation (3)

Define $\mu_t^{nj,ik}$ as the fraction of workers that relocate from labor market $nj$ to $ik$. This fraction can be interpreted as the probability that a given worker moves from labor market $nj$ to $ik$ at time $t$. This can be interpreted as the probability that the expected utility of moving to $ik$ is greater than $nj$.

$$\begin{aligned}
\mu_t^{nj,ik} &= \Pr\left(\frac{\beta V_{t+1}^{ik} - \tau^{nj,ik}}{v} + \varepsilon_t^{ik} \geq \max_{mh \neq ik}\left\{\frac{\beta V_{t+1}^{mh} - \tau^{nj,mh}}{v} + \varepsilon_t^{mh}\right\}\right) \\
&= \int_{-\infty}^{\infty} f\left(\varepsilon_t^{ik}\right) \prod_{mh \neq ik} F\left(\bar{\varepsilon}_t^{ik,mh} + \varepsilon_t^{ik}\right) d\varepsilon_t^{ik} \\
&= \int_{-\infty}^{\infty} e^{\left(-\varepsilon_t^{ik} - \gamma\right)} e^{\left(-e^{\left(-\varepsilon_t^{ik} - \gamma\right)} \sum_{ik} e^{\left(-\bar{\varepsilon}_t^{ik,mh}\right)}\right)} d\varepsilon_t^{ik} \\
&= \exp\left(-\lambda_t^{ik}\right) \int_{-\infty}^{\infty} \exp\left(-\tilde{y}_t^{ik} - \exp\left(-\tilde{y}_t^{ik}\right)\right) d\tilde{y}_t^{ik} \\
&= \exp\left(-\log\left(\sum_{mh} e^{\left(-\bar{\varepsilon}_t^{ik,mh}\right)}\right)\right) \\
&= \exp\left(\log\left(\frac{\exp\left(\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)\right)^{\frac{1}{\nu}}}{\sum_{mh} \exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{\frac{1}{\nu}}}\right)\right) \\
&= \frac{\exp\left(\beta V_{t+1}^{ik} - \tau^{nj,ik}\right)^{\frac{1}{\nu}}}{\sum_{mh} \exp\left(\beta V_{t+1}^{mh} - \tau^{nj,mh}\right)^{\frac{1}{\nu}}}
\end{aligned}$$

The labor market distribution can be though of as a Markov process. Given that $\mu_t^{nj,ik}$ is a transition matrix which represents the probability of being going from state $nj$ to state $ik$ at period $t+1$, we can write our labor distribution as

$$L_{t+1}^{nj} = \sum_{ik} \mu_t^{ik,nj} L_t^{ik}$$

## 2.3 Proposition 2

"Conditional on an initial allocation of the economy, $(L_0, \pi_0, X_0, \mu_{-1})$, given an anticipated sequence of changes in fundamentals, $\left\{\dot{\Theta}\right\}_{t=1}^{\infty}$, with $\lim_{t\to\infty} \dot{\Theta}_t = 1$, the solution to the sequential equilibrium in time differences does not require information on the level of the fundamentals, $\{\Theta_t\}_{t=0}^{\infty}$ or $\bar{\Theta}$ , and solves the following system of nonlinear equations:

$$\begin{aligned}
\mu_{t+1}^{nj,ik} &= \frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{mh} \mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}} \\
\dot{u}_{t+1}^{nj} &= \dot{\omega}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)\left(\sum_{ik} \mu^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}
\end{aligned}$$

$$L_{t+1}^{nj} = \sum_{ik} \mu_t^{ik,nj} L_t^{ik}$$

for all $j, n, i$, and $k$ at each $t$ where $\left\{\dot{\omega}^{nj}\left(\dot{L}_t, \dot{\Theta}_t\right)\right\}_{n=1,j=0,t=1}^{N,J,\infty}$, is the solution to the temporary equilibrium given $\left\{\dot{L}_t, \dot{\Theta}_t\right\}_{t=1}^{\infty}$ "

This proposition gives part of the algorithm required to solve the DP in changes. It's usefulness comes from that fact that we don't need to know the fundamentals of each time period in levels $\left(\Theta_t \equiv (A_t, \kappa_t), \bar{\Theta} \equiv (\mathcal{Y}, H, b)\right)$.

The transition matrix at $t = 0$ is given by

$$\mu_0^{nj,ik} = \frac{\exp\left(\beta V_1^{ik} - \tau^{nj,ij}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_1^{mh} - \tau^{nj,mh}\right)^{1/\nu}}$$

Now dividing by the transition matrix at $t = -1$, which are just the migration flows taken from the data, we get

$$\begin{aligned}
\frac{\mu_0^{nj,ik}}{\mu_{-1}^{nj,ik}} &= \frac{\frac{\exp\left(\beta V_1^{ik} - \tau^{nj,ij}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_1^{mh} - \tau^{nj,mh}\right)^{1/\nu}}}{\frac{\exp\left(\beta V_0^{ik} - \tau^{nj,ij}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_0^{mh} - \tau^{nj,mh}\right)^{1/\nu}}} \\
&= \frac{\exp\left(\beta V_1^{ik} - \beta V_0^{ik}\right)^{1/\nu}}{\frac{\sum_{mh} \left(\beta V_1^{mh} - \tau^{nj,mh}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_0^{mh} - \tau^{nj,mh}\right)^{1/\nu}}} \\
&= \frac{\exp\left(\beta V_1^{ik} - \beta V_0^{ik}\right)^{1/\nu}}{\sum_{mh} \frac{\exp\left(\beta V_1^{mh} - \tau^{nj,mh}\right)^{1/\nu}}{\exp\left(\beta V_0^{mh} - \tau^{nj,mh}\right)^{1/\nu}} \frac{\exp\left(\beta V_0^{mh} - \tau^{nj,mh}\right)^{1/\nu}}{\sum_{mh} \exp\left(\beta V_0^{mh} - \tau^{nj,mh}\right)^{1/\nu}}} \\
&= \frac{\exp\left(V_1^{ik} - V_0^{ik}\right)^{\beta/\nu}}{\sum_{mh} \mu_{-1}^{nj,mh} \exp\left(V_1^{mh} - V_0^{mh}\right)^{\beta/\nu}} \\
\dot{u}_1^{nj} &= \exp\left(V_1^{nj} - V_0^{nj}\right) \\
\mu_0^{nj,ik} &= \frac{\mu_{-1}^{nj,ik} \left(\dot{u}_t^{ik}\right)^{\beta/\nu}}{\sum_{mh} \mu_{-1}^{nj,mh} \left(\dot{u}_t^{mh}\right)^{\beta/\nu}} \\
V_1^{nj} - V_0^{nj} &= U\left(C_1^{nj}\right) - U\left(C_0^{nj}\right) + \nu \log\left(\frac{\beta \sum_{ik} \exp\left(\beta V_2^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\beta \sum_{ik} \exp\left(\beta V_1^{ik} - \tau^{nj,ik}\right)^{1/\nu}}\right) \\
&= a + \nu \log\left(\sum_{ik} \frac{\exp\left(\beta V_2^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\exp\left(\beta V_1^{ik} - \tau^{nj,ik}\right)^{1/\nu}} \frac{\exp\left(\beta V_1^{ik} - \tau^{nj,ik}\right)^{1/\nu}}{\sum_{ik} \exp\left(\beta V_1^{ik} - \tau^{nj,ik}\right)^{1/\nu}}\right) \\
&= U\left(C_1^{nj}\right) - U\left(C_0^{nj}\right) + \nu \log\left(\sum_{ik} \mu_0^{nj,ik} \exp\left(V_2^{ik} - V_1^{ik}\right)^{\beta/\nu}\right) \\
&= \log\left(\frac{\dot{w}_t^{nj}}{\dot{P}_t^{nj}}\right) + \nu \log\left(\sum_{ik} \mu_0^{nj,ik} \exp\left(V_2^{ik} - V_1^{ik}\right)^{\beta/\nu}\right) \\
\exp\left(V_1^{nj} - V_0^{nj}\right) &= \exp\left(\log\left(\frac{\dot{w}_t^{nj}}{\dot{P}_t^{nj}}\right) + \nu \log\left(\sum_{ik} \mu_0^{nj,ik} \exp\left(V_2^{ik} - V_1^{ik}\right)^{\beta/\nu}\right)\right) \\
\dot{u}_1^{nj} &= \frac{\dot{w}_1^{nj}}{\dot{P}_1^{nj}} \left(\sum_{ik} \mu_0^{nj,ik} \left(\dot{u}_2^{ik}\right)^{\beta/\nu}\right)^{\nu} \\
\dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right) &\equiv \frac{\dot{w}_1^{nj}}{\dot{P}_1^{nj}}
\end{aligned}$$

$$\dot{u}_{t+1}^{nj} = \dot{\omega}^{nj}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)\left(\sum_{ik} \mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

# 3 Dynamic Programming II

Now, we need to solve for the counterfactual sequential equilibrium given by proposition 3. The system of nonlinear equations is

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\dot{\mu}_{t+1}^{nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{\prime nj.mh}\dot{\mu}_{t+1}^{nj,mh}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

$$\hat{u}_{t+1}^{nj} = \hat{\omega}\left(\dot{L}_{t+1}, \dot{\Theta}_{t+1}\right)\left(\sum_{ik}\mu_t^{\prime nj,ik}\dot{\mu}_{t+1}^{nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

$$L_{t+1}^{\prime nj} = \sum_{ik}\mu_t^{\prime ik,nk}L_t^{\prime ik}$$

where $\left\{\hat{\omega}^{nj}\left(\hat{L}_t, \hat{\Theta}_t\right)\right\}_{n=1,j=1,t=1}^{N,J,\infty}$ is the solution to the temporary equilibrium described in the following section at each $t$.

Now to show why each is true starting with the migration probabilities.

$$\frac{\mu_{t+1}^{\prime nj,ik}}{\mu_{t+1}^{nj,ik}} = \frac{\dfrac{\mu_t^{\prime nj,ik}\left(\dot{u}_{t+2}^{\prime ik}\right)^{\beta/\nu}}{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}}{\dfrac{\sum_{mh}\mu_t^{\prime nj.mh}\left(\dot{u}_{t+2}^{\prime mh}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dfrac{\mu_{t+1}^{nj,ik}}{\mu_t^{nj,ik}}}{\sum_{mh}\dfrac{\mu_t^{\prime nj.mh}\left(\dot{u}_{t+2}^{\prime mh}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\dfrac{\mu_t^{\prime nj.mh}\left(\dot{u}_{t+2}^{\prime mh}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\dfrac{\mu_t^{\prime nj.mh}\left(\dot{u}_{t+2}^{\prime mh}\right)^{\beta/\nu}\frac{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\left(\frac{\mu_t^{\prime nj.mh}}{\mu_t^{nj,mh}}\right)\dfrac{\mu_t^{nj,mh}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

Recall,

$$\mu_{t+1}^{nj,ik} = \frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

$$\sum_{mh}\mu_t^{nj.mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu} = \frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\mu_{t+1}^{nj,ik}}$$

Substituting this back into denominator,

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\left(\frac{\mu_t^{\prime nj.mh}}{\mu_t^{nj,mh}}\right)\frac{\mu_t^{nj,mh}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}}{\frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\mu_{t+1}^{nj,ik}}}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\left(\frac{\mu_t^{\prime nj.mh}}{\mu_t^{nj,mh}}\right)\frac{\mu_{t+1}^{nj,ik}}{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}\frac{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}\mu_t^{nj,mh}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\left(\frac{\mu_t^{\prime nj.mh}}{\mu_t^{nj,mh}}\right)\mu_{t+1}^{nj,ik}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

$$\mu_{t+1}^{\prime nj,ik} = \frac{\mu_t^{\prime nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\dot{\mu}_{t+1}^{nj,ik}}{\sum_{mh}\mu_t^{\prime nj.mh}\dot{\mu}_{t+1}^{nj,mh}\left(\hat{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

**Typo on page 797 line 1. Supposed to be $\hat{u}$ not $\dot{u}$**
Solving for values in time differences,

$$\frac{\dot{u}_{t+1}^{\prime nj}}{\dot{u}_{t+1}^{nj}} = \frac{\dot{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)^{\prime}\left(\sum_{ik}\mu_t^{\prime nj,ik}\left(\dot{u}_{t+2}^{\prime ik}\right)^{\beta/\nu}\right)^{\nu}}{\dot{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}}$$

$$\hat{u}_{t+1}^{nj} = \hat{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\frac{\mu_t^{\prime nj,ik}\left(\dot{u}_{t+2}^{\prime ik}\right)^{\beta/\nu}\frac{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}{\mu_t^{nj,mh}\left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}}{\sum_{ik}\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}\right)^{\nu}$$

$$\hat{u}_{t+1}^{nj} = \hat{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\left(\frac{\mu_t^{\prime nj,ik}}{\mu_t^{nj,ik}}\right)\frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{ik}\mu^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

$$= \hat{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\left(\frac{\mu_t^{\prime nj,ik}}{\mu_t^{nj,ik}}\right)\frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\frac{\mu_t^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\mu_{t+1}^{nj,ik}}}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

$$= \hat{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\left(\frac{\mu_t^{\prime nj,ik}}{\mu_t^{nj,ik}}\right)\mu_{t+1}^{nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

$$\hat{u}_{t+1}^{nj} = \hat{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\mu_t^{\prime nj,ik}\dot{\mu}_{t+1}^{nj,ik}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

## 3.1 Solution algorithm

To solve the DP problem, we need to know the initial distribution of workers and the transition matrix at $t = 1$. This is taken from the data. The first step is to guess the vector for expected lifetime changes in utility in each market, $\bar{\dot{u}}$. Given this guess, we can use it to solve for our transition matrices for each $t$ and the evolution of labor employment. We use these values $\mathcal{D}\left(\bar{\dot{u}},\bar{\mu},\bar{L}\right)$ to solve for our static, temporary equilibrium at each $t$.

# 4 Temporary equilibrium

Solving the temporary equilibrium given our guesses to the DP problem is straightforward.

The intermediate producer problem is given by

$$\min_{\left\{l_t^{nj}, h_t^{nj}, M_t^{nk,nj}\right\}} w_t^{nj} l_n^{nj} + r_t^{nj} h_t^{nj} + P_t^{nk} M_t^{nj,nk}$$

$$\text{s.t. } 1 = z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}$$

The Lagrangian is given by

$$\mathcal{L} = w_t^{nj} l_t^{nj} + r_t^{nj} h_t^{nj} + P_t^{nk} M_t^{nj,nk} - \lambda \left( z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}} - 1 \right)$$

$$\frac{\partial L}{\partial l_t^{nj}} : w_t^{nj} = \lambda \gamma^{nj} z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}-1} \left[ \left(1-\xi^n\right) A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{-\xi^n} \right] \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}} \tag{4}$$

$$\frac{\partial L}{\partial h_t^{nj}} : r_t^{nj} = \lambda \gamma^{nj} z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}-1} \left[ \xi^n A_t^{nj} \left(h_t^{nj}\right)^{\xi^n - 1} \left(l_t^{nj}\right)^{1-\xi^n} \right] \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}$$

$$\frac{\partial L}{\partial M_t^{nk,nj}} : P_t^{nk} = \lambda z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}} \gamma^{nk,nj} \left(M_t^{nj,nk}\right)^{-1} \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}$$

Solving the FOCs, we can substitute back into equation (4) to solve the unit cost of an input bundle.

$$\frac{w_t^{nj}}{r_t^{nj}} = \frac{\lambda \gamma^{nj} z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}-1} \left[ \left(1-\xi^n\right) A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{-\xi^n} \right] \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}}{\lambda \gamma^{nj} z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}-1} \left[ \xi^n A_t^{nj} \left(h_t^{nj}\right)^{\xi^n-1} \left(l_t^{nj}\right)^{1-\xi^n} \right] \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}}$$

$$l_t^{nj} = h_t^{nj} \frac{r_t^{nj}}{w_t^{nj}} \frac{1-\xi^n}{\xi^n}$$

$$\frac{r_t^{nj}}{P_t^{nk}} = \frac{\lambda \gamma^{nj} z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}-1} \left[ \xi^n A_t^{nj} \left(h_t^{nj}\right)^{\xi^n-1} \left(l_t^{nj}\right)^{1-\xi^n} \right] \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}}{\lambda z^{nj} \left( A_t^{nj} \left(h_t^{nj}\right)^{\xi^n} \left(l_t^{nj}\right)^{1-\xi^n} \right)^{\gamma^{nj}} \gamma^{nk,nj} \left(M_t^{nj,nk}\right)^{-1} \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)^{\gamma^{nj,nk}}}$$

$$M_t^{nj,nk} = \frac{r_t^{nj}}{P_t^{nk}} \frac{\gamma^{nj,nk}}{\gamma^{nj}} \frac{1}{\xi^n} h_t^{nj}$$

$$\lambda^{-1} = z^{nj} \left(A_t^{nj}\right)^{\gamma^{nj}} \gamma^{nj} \left(h_t^{nj}\right)^{\gamma^{nj}-1} \left(\frac{1}{w_t^{nj}}\right)^{(1-\xi^n)\gamma^{nj}} \left(r_t^{nj}\right)^{(1-\xi^n)\gamma^{nj}-1} \left(1-\xi^n\right)^{(1-\xi^n)\gamma^{nj}} \left(\frac{1}{\xi^n}\right)^{(1-\xi^n)\gamma^{nj}-1} \prod_{k=1}^{J} \left(M_t^{nj,nk}\right)$$

$$\lambda^{-1} = z^{nj} \left(A_t^{nj}\right)^{\gamma^{nj}} \left[ \left(\frac{1}{w_t^{nj}}\right)^{(1-\xi^n)} \left(\frac{1}{r_t^{nj}}\right)^{\xi^n} \right]^{\gamma^{nj}} \prod_{k=1}^{J} \left(\frac{1}{P_t^{nk}}\right)^{\gamma^{nk,nj}} \left(1-\xi^n\right)^{(1-\xi^n)\gamma^{nj}} \left(\xi^n\right)^{\gamma^{nj}\xi^n} \prod_{k=1}^{J} \left(\gamma^{nj}\right)^{\gamma^{nj}} \left(\gamma^{nj,nk}\right)^{\gamma^{nj,nk}}$$

$$\left(B^{nj}\right)^{-1} = \left(1-\xi^n\right)^{(1-\xi^n)\gamma^{nj}} \left(\xi^n\right)^{\gamma^{nj}\xi^n} \prod_{k=1}^{J} \left(\gamma^{nj}\right)^{\gamma^{nj}} \left(\gamma^{nj,nk}\right)^{\gamma^{nj,nk}}$$

$$\lambda = \frac{\left(B^{nj}\right) \left[ \left(r_t^{nj}\right)^{\xi^n} \left(w_t^{nj}\right)^{(1-\xi^n)} \right]^{\gamma^{nj}} \prod_{k=1}^{J} \left(P_t^{nk}\right)^{\gamma^{nj,nk}}}{\left( z^{nj} \left(A_t^{nj}\right)^{\gamma^{nj}} \right)}$$

where the unit price of an input bundle is given by

$$x_t^{nj} = \left(B^{nj}\right)\left[\left(r_t^{nj}\right)^{\xi^n}\left(w_t^{nj}\right)^{(1-\xi^n)}\right]^{\gamma^{nj}}\prod_{k=1}^{J}\left(P_t^{nk}\right)^{\gamma^{nj,nk}}$$

and the unit cost of an intermediate good is just the shadow price, $\lambda$. Since we assume perfect competition, intermediate producers price at marginal cost. Therefore, the unit cost of intermediate good $z_t^{nj}$ is

$$\frac{x_t^{nj}}{\left(z^{nj}\left(A_t^{nj}\right)^{\gamma^{nj}}\right)}$$

Producers around the world search for the lowest cost supplier and therefore, the price paid is determined by the minimum unit cost across regions

$$p_t^{nj} = \min\left\{\frac{\kappa_t^{nj,ij}x_t^{nj}}{z^{nj}\left(A_t^{nj}\right)^{\gamma^{nj}}}\right\}$$

Local suppliers demand intermediate varieties from the lowest costs suppliers across all regions. Intermediates are used to produce local sectoral aggregate goods, $Q_t^{nj}$ at price $P_t^{nj}$. The demand for varieties is given by $\tilde{q}_t^{nj}$ and we know from above that the price paid for this good is given by $p_t^{nj}$. Therefore, producers seek to minimize costs subject to one unit of production. The problem can be characterized by,

$$\begin{aligned} &\min \int p_t^{nj}\left(z_t^j\right)\tilde{q}_t^{nj}\left(z_t^j\right)d\phi \\ &\text{s.t. } Q_t^{nj} = 1 \end{aligned}$$

where the aggregate sectoral good is CES

$$Q_t^{nj} = \left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}}d\phi^j\left(z^j\right)\right)^{\eta^{nj}/(\eta^{nj}-1)}$$

The Lagrangian is given by,

$$\begin{aligned}
\mathcal{L}\left(q_t^{nj},\lambda\right) &= \int p_t^{nj}\left(z_t^j\right)\tilde{q}_t^{nj}\left(z_t^j\right)d\phi - \lambda\left(1-\left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}}d\phi^j\left(z^j\right)\right)^{\eta^{nj}/(\eta^{nj}-1)}\right) \\
\frac{\partial\mathcal{L}}{\partial\tilde{q}_t^{nj}} : p_t^{nj} &= \lambda\frac{\eta^{nj}}{\eta^{nj}-1}\left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}}d\phi^j\left(z^j\right)\right)^{\frac{1}{(\eta^{nj}-1)}}\left[1-\frac{1}{\eta^{nj}}\right]\left(\tilde{q}_t^{nj}\left(z^j\right)^{-\frac{1}{\eta^{nj}}}\right) \\
\frac{\partial\mathcal{L}}{\partial\lambda} : 1 &= \left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}}d\phi^j\left(z^j\right)\right)^{\eta^{nj}/(\eta^{nj}-1)} \\
\left(p_t^{nj}\right)^{\eta^{nj}} &= \lambda^{\eta^{nj}}\left(\int\left(\tilde{q}_t^{nj}\left(z^j\right)\right)^{1-1/\eta^{nj}}d\phi^j\left(z^j\right)\right)^{\frac{\eta^{nj}}{(\eta^{nj}-1)}}\left(\tilde{q}_t^{nj}\left(z^j\right)^{-1}\right) \\
\left(p_t^{nj}\right)^{\eta^{nj}} &= \lambda^{\eta^{nj}}Q_t^{nj}\left(\tilde{q}_t^{nj}\left(z^j\right)^{-1}\right) \\
\tilde{q}_t^{nj}\left(z^j\right) &= \left(\frac{\lambda}{p_t^{nj}}\right)^{\eta^{nj}} \\
\tilde{q}_t^{nj}\left(z^j\right) &= \left(\frac{p_t^{nj}}{\lambda}\right)^{-\eta^{nj}}
\end{aligned}$$

Substituting this into our shadow cost,

$$1 = \left( \int \left( \tilde{q}_t^{nj}\left(z^j\right) \right)^{1-1/\eta^{nj}} d\phi^j\left(z^j\right) \right)^{\eta^{nj}/(\eta^{nj}-1)}$$

$$= \lambda^{\eta^{nj}} \left( \int \left( \left( p_t^{nj} \right)^{-(\eta^{nj}-1)} \right) d\phi^j\left(z^j\right) \right)^{\eta^{nj}/(\eta^{nj}-1)}$$

$$\lambda = \left( \int \left( \left( p_t^{nj} \right)^{1-\eta^{nj}} \right) d\phi^j\left(z^j\right) \right)^{1/(\eta^{nj}-1)}$$

Since we assume competition, we know that producers price at marginal cost, therefore we know that the price of our local sectoral aggregate goods are given by

$$P_t^{nj} = \left( \int \left( \left( p_t^{nj} \right)^{1-\eta^{nj}} \right) d\phi^j\left(z^j\right) \right)^{1/(\eta^{nj}-1)}$$

Using properties of Fréchet distribution, we can rewrite aggregate prices as

$$P_t^{nj} = \Gamma^{nj} \left( \sum_{i \in N} \left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}} \right)^{-\frac{1}{\theta^j}}$$

Define total expenditures in country $n$ on sector $j$ is defined by $X_t^{nj} = P_t^{nj} Q_t^{nj}$. We can say that the share of expenditures from market $nj$ to market $ik$ is given by, $\pi_t^{nj,ij} = \frac{X_t^{nj,ij}}{X_t^{nj}}$. Returning to our cost minimization problem and using properties of the Fréchet distribution, we can say the following, **PROVE THIS**

$$\left( p_t^{nj} \right)^{\eta^{nj}} = \left( P_t^{nj} \right)^{\eta^{nj}} Q_t^{nj} \left( \tilde{q}_t^{nj}\left(z^j\right) \right)^{-1}$$

$$\tilde{q}_t^{nj}\left(z^j\right) = \left( \frac{p_t^{nj}}{P_t^{nj}} \right)^{-\eta^{nj}} Q_t^{nj}$$

$$p_t^{nj}\left(z^j\right) \tilde{q}_t^{nj}\left(z^j\right) = \left( \frac{p_t^{nj}}{P_t^{nj}} \right)^{1-\eta^{nj}} P_t^{nj} Q_t^{nj}$$

$$\int p_t^{nj}\left(z^j\right) \tilde{q}_t^{nj}\left(z^j\right) d\tilde{\phi} = \left( \frac{1}{P_t^{nj}} \right)^{1-\eta^{nj}} P_t^{nj} Q_t^{nj} \int \left( p_t^{nj} \right)^{1-\eta^{nj}} d\tilde{\phi}$$

$$X_t^{nj,ij} = \left( \frac{\Gamma^{nj} \left( \left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}} \right)}{\Gamma^{nj} \left( \sum_{i \in N} \left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}} \right)} \right) X_t^{nj}$$

$$X_t^{nj,ij} = \left( \frac{\left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}}}{\sum_{i \in N} \left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}}} \right) X_t^{nj}$$

$$\pi_t^{nj,ij} = \frac{\left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}}}{\sum_{i \in N} \left( x_t^{ij} \kappa_t^{nj,ij} \right)^{-\theta^j} \left( A_t^{ij} \right)^{\theta^j \gamma^{ij}}}$$

Now for market clearing and unbalanced trade. Total expenditures are defined as the sum of expenditures on intermediates and final goods,

$$X_t^{nj} = \sum_{ik} \gamma^{nk,nj} \pi_t^{ik,nk} X_t^{ik} + \alpha^j \left( \sum_k w_t^{nk} L_t^{nk} + \iota^n \mathcal{X}_t \right) \tag{5}$$

where $\iota^n \mathcal{X}_t$ represents the share of the global portfolio received by renters. The total revenue of the global portfolio is given by $\mathcal{X}_t = \sum_{ik} r_t^{ik} H_t^{ik}$. Recall from our FOC of our cost-minimization, we had the following factor price relationship,

$$\frac{w_t^{nj} L_t^{nj}}{1-\xi^n} = \frac{r_t^{nj} H_t^{nj}}{\xi^n} \tag{6}$$
$$\frac{r_t^{nj} H_t^{nj}}{\xi^n \gamma^{nj}} = \frac{M_t^{nj,nk} P_t^{nk}}{\gamma^{nj,nk}}$$
$$\frac{w_t^{nj} L_t^{nj}}{(1-\xi^n)\gamma^{nj}} = \frac{M_t^{nj,nk} P_t^{nk}}{\gamma^{nj,nk}}$$

We can rearrange our relationship between wages/rents and materials and substitute back into equation (6) to derive our factor market clearing conditions.

$$r_t^{nj} H_t^{nj} \gamma^{nk,nj} = \xi^n \gamma^{nj} M_t^{nj,nk} P_t^{nk}$$
$$r_t^{nj} H_t^{nj} \sum_k \gamma^{nj,nk} = \xi^n \gamma^{nj} \sum_k M_t^{nj,nk} P_t^{nk}$$
$$r_t^{nj} H_t^{nj} \left(1-\gamma^{nj}\right) = \xi^n \gamma^{nj} \sum_i \sum_k \gamma^{nk,nj} \pi_t^{ik,nk} X_t^{ik}$$
$$r_t^{nj} H_t^{nj} = \xi^n \gamma^{nj} \frac{\sum_i \sum_k \gamma^{nk,nj} \pi_t^{ik,nk} X_t^{ik}}{(1-\gamma^{nj})}$$
$$r_t^{nj} H_t^{nj} = \xi^n \gamma^{nj} \sum_i \pi_t^{ij,nj} X_t^{ij}$$

$$w_t^{nj} L_t^{nj} = r_t^{nj} H_t^{nj} \frac{1-\xi^n}{\xi^n}$$
$$w_t^{nj} L_t^{nj} = \left(\xi^n \gamma^{nj} \sum_i \pi_t^{ij,nj} X_t^{ij}\right) \frac{1-\xi^n}{\xi^n}$$
$$w_t^{nj} L_t^{nj} = (1-\xi^n)\gamma^{nj} \sum_i \pi_t^{ij,nj} X_t^{ij}$$
$$L_t^{nj} = \frac{(1-\xi^n)\gamma^{nj} \sum_i \pi_t^{ij,nj} X_t^{ij}}{w_t^{nj}}$$

and similarly for structures,

$$H_t^{nj} = \frac{\xi^n \gamma^{nj} \sum_i \pi_t^{ij,nj} X_t^{ij}}{r_t^{nj}}$$

## 4.1 Time differences

Now, we solve the model in time differences.

Input costs:

$$\frac{x_{t+1}^{nj}}{x_t^{nj}} = \frac{(B^{nj}) \left[\left(r_{t+1}^{nj}\right)^{\xi^n} \left(w_{t+1}^{nj}\right)^{(1-\xi^n)}\right]^{\gamma^{nj}} \prod_{k=1}^J \left(P_{t+1}^{nk}\right)^{\gamma^{nj,nk}}}{(B^{nj}) \left[\left(r_t^{nj}\right)^{\xi^n} \left(w_t^{nj}\right)^{(1-\xi^n)}\right]^{\gamma^{nj}} \prod_{k=1}^J \left(P_t^{nk}\right)^{\gamma^{nj,nk}}}$$

$$\frac{x_{t+1}^{nj}}{x_t^{nj}} = \frac{(B^{nj}) \left[\left(w_{t+1}^{nj} L_{t+1}^{nj} \frac{\xi^n}{(1-\xi^n)}\right)^{\xi^n} \left(w_{t+1}^{nj}\right)^{(1-\xi^n)}\right]^{\gamma^{nj}} \prod_{k=1}^J \left(P_{t+1}^{nk}\right)^{\gamma^{nj,nk}}}{(B^{nj}) \left[\left(w_t^{nj} L_t^{nj} \frac{\xi^n}{(1-\xi^n)}\right)^{\xi^n} \left(w_t^{nj}\right)^{(1-\xi^n)}\right]^{\gamma^{nj}} \prod_{k=1}^J \left(P_t^{nk}\right)^{\gamma^{nj,nk}}}$$

$$\dot{x}_{t+1}^{nj} = \left(\dot{L}_{t+1}^{nj}\right)^{\xi^n \gamma^{nj}} \left(\dot{w}_{t+1}^{nj}\right)^{\gamma^{nj}} \prod_{k \in J} \left(\dot{P}_{t+1}^{nk}\right)^{\gamma^{nj,nk}}$$

$$\ln\left(\dot{x}_{t+1}^{nj}\right) = \xi^n \gamma^{nj} \ln\left(\dot{L}_{t+1}^{nj}\right) + \gamma^{nj} \ln\left(\dot{w}_{t+1}^{nj}\right) + \sum_{k \in J} \gamma^{nj,nk} \ln\left(\dot{P}_{t+1}^{nk}\right)$$

Sectoral price index:

$$\frac{P_{t+1}^{nj}}{P_t^{nj}} = \frac{\Gamma^{nj}\left(\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\right)^{-\frac{1}{\theta^j}}}{\Gamma^{nj}\left(\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)^{-\frac{1}{\theta^j}}}$$

$$\frac{P_{t+1}^{nj}}{P_t^{nj}} = \left[\frac{\left(\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\right)}{\left(\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)}\right]^{-\frac{1}{\theta^j}}$$

$$\frac{P_{t+1}^{nj}}{P_t^{nj}} = \left[\frac{\left(\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\right)}{\left(\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)}\right]^{-\frac{1}{\theta^j}}$$

$$\frac{P_{t+1}^{nj}}{P_t^{nj}} = \left[\sum_{i \in N}\frac{\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\left(\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}\right)}\right]^{-\frac{1}{\theta^j}}$$

$$\dot{P}_{t+1}^{nj} = \left[\sum_{i \in N}\pi_t^{nj,ij}\left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^j}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\right]^{-\frac{1}{\theta^j}}$$

Expenditure shares:

$$\frac{\pi_{t+1}^{nj,ij}}{\pi_t^{nj,ij}} = \frac{\frac{\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}}{\frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}}$$

$$\frac{\pi_{t+1}^{nj,ij}}{\pi_t^{nj,ij}} = \frac{\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\frac{\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}$$

$$\frac{\pi_{t+1}^{nj,ij}}{\pi_t^{nj,ij}} = \left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^j}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\left(\frac{\sum_{i \in N}\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\right)^{-1}$$

$$\frac{\pi_{t+1}^{nj,ij}}{\pi_t^{nj,ij}} = \left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^j}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\left(\sum_{i \in N}\frac{\left(x_{t+1}^{ij}\kappa_{t+1}^{nj,ij}\right)^{-\theta^j}\left(A_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}}{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\frac{\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}{\sum_{i \in N}\left(x_t^{ij}\kappa_t^{nj,ij}\right)^{-\theta^j}\left(A_t^{ij}\right)^{\theta^j\gamma^{ij}}}\right)^{-1}$$

$$\frac{\pi_{t+1}^{nj,ij}}{\pi_t^{nj,ij}} = \left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^j}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^j\gamma^{ij}}\left(\dot{P}_{t+1}^{nj}\right)^{\theta^j}$$

$$\pi_{t+1}^{nj,ij}=\pi_{t}^{nj,ij}\left(\frac{\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}}{\dot{P}_{t+1}^{nj}}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^{j}\gamma^{ij}}$$

Total expenditures:

$$X_{t+1}^{nj}=\sum_{ik}\gamma^{nk,nj}\pi_{t+1}^{ik,nk}X_{t+1}^{ik}+\alpha^{j}\left(\sum_{k}\dot{w}_{t+1}^{nk}\dot{L}_{t+1}^{nk}w_{t}^{nk}L_{t}^{nk}+\iota^{n}\mathcal{X}_{t+1}\right)$$

$$\mathcal{X}_{t+1}=\sum_{ik}\frac{\xi^{i}}{(1-\xi^{i})}\dot{w}_{t+1}^{ij}\dot{L}_{t+1}^{ij}w_{t}^{ij}L_{t}^{ij}$$

Labor market clearing

$$\dot{w}_{t+1}^{nj}\dot{L}_{t+1}^{nj}w_{t}^{nj}L_{t}^{nj}=\gamma^{nj}\left(1-\xi^{n}\right)\sum_{i}\pi_{t}^{ij,nj}X_{t}^{ij}$$

## 4.2 Counterfactual time differences

Now, we need to rewrite the equilibrium conditions in counterfactual changes over time.

Starting with the dynamic programming problem,

$$\mu_{t}^{\prime nj,ik}=\frac{\mu_{t-1}^{\prime nj,ik}\left(\dot{u}_{t+1}^{\prime ik}\right)^{\beta/\nu}}{\sum_{mh}\mu_{t-1}^{\prime nj.mh}\left(\dot{u}_{t+1}^{\prime mh}\right)^{\beta/\nu}}$$

$$\mu_{t}^{\prime nj,ik}=\frac{\mu_{t-1}^{\prime nj,ik}\left(\hat{u}_{t+1}^{ik}\right)^{\beta/\nu}\left(\dot{u}_{t+1}^{ik}\right)^{\beta/\nu}}{\sum_{mh}\mu_{t-1}^{\prime nj.mh}\left(\hat{u}_{t+1}^{mh}\right)^{\beta/\nu}\left(\dot{u}_{t+1}^{mh}\right)^{\beta/\nu}}$$

$$\dot{u}_{t+1}^{nj}=\dot{\omega}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\mu^{nj,ik}\left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)$$

$$\hat{u}_{t+1}^{nj}=\hat{\omega}^{nj}\left(\dot{L}_{t+1},\dot{\Theta}_{t+1}\right)\left(\sum_{ik}\mu_{t}^{\prime nj,ik}\dot{\mu}_{t+1}^{nj,ij}\left(\hat{u}_{t+2}^{ik}\right)^{\beta/\nu}\right)^{\nu}$$

Input cost bundle

$$\frac{\dot{x}_{t+1}^{\prime nj}}{\dot{x}_{t+1}^{nj}}=\frac{\left(\dot{L}_{t+1}^{\prime nj}\right)^{\xi^{n}\gamma^{nj}}\left(\dot{w}_{t+1}^{\prime nj}\right)^{\gamma^{nj}}\prod_{k\in J}\left(\dot{P}_{t+1}^{\prime nk}\right)^{\gamma^{nj,nk}}}{\left(\dot{L}_{t+1}^{nj}\right)^{\xi^{n}\gamma^{nj}}\left(\dot{w}_{t+1}^{nj}\right)^{\gamma^{nj}}\prod_{k\in J}\left(\dot{P}_{t+1}^{nk}\right)^{\gamma^{nj,nk}}}$$

$$\hat{x}_{t+1}^{nj}=\left(\hat{L}_{t+1}^{nj}\right)^{\xi^{n}\gamma^{nj}}\left(\hat{w}_{t+1}^{nj}\right)^{\gamma^{nj}}\prod_{k\in J}\left(\hat{P}_{t+1}^{nj}\right)^{\gamma^{nj,nk}}$$

$$\ln\left(\hat{x}_{t+1}^{nj}\right)=\xi^{n}\gamma^{nj}\ln\left(\hat{L}_{t+1}^{nj}\right)+\gamma^{nj}\ln\left(\hat{w}_{t+1}^{nj}\right)+\sum_{k\in J}\gamma^{nj,nk}\ln\left(\hat{P}_{t+1}^{nj}\right)$$

Sectoral price index

$$\frac{\dot{P}_{t+1}^{\prime nj}}{\dot{P}_{t+1}^{nj}}=\frac{\left[\sum_{i\in N}\pi_{t}^{\prime nj,ij}\left(\dot{x}_{t+1}^{\prime ij}\dot{\kappa}_{t+1}^{\prime nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{\prime ij}\right)^{\theta^{j}\gamma^{ij}}\right]^{-\frac{1}{\theta^{j}}}}{\left[\sum_{i\in N}\pi_{t}^{nj,ij}\left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^{j}\gamma^{ij}}\right]^{-\frac{1}{\theta^{j}}}}$$

$$\hat{P}_{t+1}^{nj}=\frac{\left[\sum_{i\in N}\pi_{t}^{\prime nj,ij}\left(\dot{x}_{t+1}^{\prime ij}\dot{\kappa}_{t+1}^{\prime nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{\prime ij}\right)^{\theta^{j}\gamma^{ij}}\frac{\pi_{t}^{nj,ij}\left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^{j}\gamma^{ij}}}{\pi_{t}^{nj,ij}\left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^{j}\gamma^{ij}}}\right]^{-\frac{1}{\theta^{j}}}}{\left[\sum_{i\in N}\pi_{t}^{nj,ij}\left(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^{j}}\left(\dot{A}_{t+1}^{ij}\right)^{\theta^{j}\gamma^{ij}}\right]^{-\frac{1}{\theta^{j}}}}$$

$$\hat{P}^{nj}_{t+1} = \left[ \sum_{i \in N} \frac{\pi'^{nj,ij}_t \left( \dot{x}'^{ij}_{t+1} \dot{\kappa}'^{nj,ij}_{t+1} \right)^{-\theta^j} \left( \dot{A}'^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}}{\pi^{nj,ij}_t \left( \dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1} \right)^{-\theta^j} \left( \dot{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}} \frac{\pi^{nj,ij}_t \left( \dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1} \right)^{-\theta^j} \left( \dot{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}}{\left[ \sum_{i \in N} \pi^{nj,ij}_t \left( \dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1} \right)^{-\theta^j} \left( \dot{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}} \right]} \right]^{-\frac{1}{\theta^j}}$$

$$\hat{P}^{nj}_{t+1} = \left[ \sum_{i \in N} \pi'^{nj,ij}_t \dot{\pi}^{nj,ij}_t \left( \hat{x}^{ij}_{t+1} \hat{\kappa}^{nj,ij}_{t+1} \right)^{-\theta^j} \left( \hat{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}} \right]^{-\frac{1}{\theta^j}}$$

Expenditure shares

$$\frac{\pi'^{nj,ij}_{t+1}}{\pi^{nj,ij}_{t+1}} = \frac{\pi'^{nj,ij}_t \left( \frac{\dot{x}'^{ij}_{t+1} \dot{\kappa}'^{nj,ij}_{t+1}}{\dot{P}'^{nj}_{t+1}} \right)^{-\theta^j} \left( \dot{A}'^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}}{\pi^{nj,ij}_t \left( \frac{\dot{x}^{ij}_{t+1} \dot{\kappa}^{nj,ij}_{t+1}}{\dot{P}^{nj}_{t+1}} \right)^{-\theta^j} \left( \dot{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}}$$

$$\pi'^{nj,ij}_{t+1} = \pi'^{nj,ij}_t \dot{\pi}^{nj,ij}_{t+1} \left( \frac{\hat{x}^{ij}_{t+1} \hat{\kappa}^{nj,ij}_{t+1}}{\hat{P}^{nj}_{t+1}} \right)^{-\theta^j} \left( \hat{A}^{ij}_{t+1} \right)^{\theta^j \gamma^{ij}}$$

Expenditures

$$X'^{nj}_{t+1} = \sum_{ik} \gamma^{nk,nj} \pi'^{ik,nk}_{t+1} X'^{ik}_{t+1} + \alpha^j \left( \sum_k \hat{w}^{nk}_{t+1} \hat{L}^{nk}_{t+1} \dot{w}^{nk}_{t+1} \dot{L}^{nk}_{t+1} w'^{nk}_t L'^{nk}_t + \iota^n \mathcal{X}'_{t+1} \right)$$

Labor market clearing

$$\dot{w}'^{nj}_{t+1} \dot{L}'^{nj}_{t+1} w'^{nj}_t L'^{nj}_t = \gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t$$

$$\frac{\dot{w}'^{nj}_{t+1} \dot{L}'^{nj}_{t+1}}{\dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1}} \dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1} w'^{nj}_t L'^{nj}_t = \gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t$$

$$\hat{w}^{nj}_{t+1} \hat{L}^{nj}_{t+1} \dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1} w'^{nj}_t L'^{nj}_t = \gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t$$

$$\hat{w}^{nj}_{t+1} = \frac{\gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t}{\hat{L}^{nj}_{t+1} \dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1} w'^{nj}_t L'^{nj}_t}$$

$$\hat{w}^{nj}_{t+1} \left( \hat{L}^{nj}_{t+1} \right)^{\xi} = \frac{\gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t}{\dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1} w'^{nj}_t L'^{nj}_t} \left( \hat{L}^{nj}_{t+1} \right)^{\xi - 1}$$

$$\omega = \frac{\gamma^{nj} \left(1 - \xi^n\right) \sum_i \pi'^{ij,nj}_t X'^{ij}_t}{\dot{w}^{nj}_{t+1} \dot{L}^{nj}_{t+1} w'^{nj}_t L'^{nj}_t} \left( \hat{L}^{nj}_{t+1} \right)^{\xi - 1}$$

Global portfolio

$$\mathcal{X}'_{t+1} = \sum_{ik} \frac{\xi^i}{\left(1 - \xi^i\right)} \hat{w}^{ik}_{t+1} \hat{L}^{ik}_{t+1} \dot{w}^{ik}_{t+1} \dot{L}^{ik}_{t+1} w'^{ik}_t L'^{ik}_t$$

# 5 Sources


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