MATH 330

MATTHEW MURILLO

ABSTRACT. This document covers lecture, textbook, and video notes for Yale's Math 330 course taught by Sekhar Tatikonda. These are not the official class notes and therefore have not been reviewed by the professor. There could be information missing or incorrect so please be sure to validate with other sources when using.

CONTENTS

Date: Fall Semester 2022.

Part 1. Lecture Notes

1. Introduction to the course

This is probability course taught from a measure theoretic perspective. The textbooks for this class are by Pollard and Durrett. This course assumes knowledge on probability, linear algebra, multivariable calculus, and real analysis.

1.1. **Measure Theoretic Definitions.**

Definition 1. Measure Space

A measure space is defined by the following triple $(\Omega, \mathcal{F}, \mu)$ where Ω is the outcome space, $\mathcal{F}$ is a σ-filed, and μ is a measure. The measure space is the the central mathematical object of probability theory. In the first section of the course, we will spend time developing a measure theoretic notion of σ-field and a measure more generally. We will then apply these more abstract ideas to probability.

Example. Ω Sample/outcome space

Definition 2. Ω is the set from which our family of sets will be constructed. In probability theory, you can think of Ω as Nature which contains the set of all possible events.

Example. $(0,1)$, $\mathbb{R}^n$, $\{0,1,2,3,4,...,n\}$, etc

Definition 3. $\mathcal{F}$ σ-field, (σ-algebra)

A σ-field of measurable sets, measurable meaning we can define a measure on the set, is a set of sets, or a family of sets such that $\mathcal{F} \subseteq 2^{\Omega}$. $\mathcal{F}$ has more defining properties which we will cover in greater detail later. In probability, think of the sets in our σ-field $\mathcal{F}$ as the events which we observe.

We also call $(\Omega, \mathcal{F})$ a measurable space. Given Ω and $\mathcal{F}$ we can now define a measure μ

Definition 4. $\mu : \mathcal{F} \to \mathbb{R}^+$ is a measure

We call μ probability measure if $\mu(\Omega) = 1$. Essentially, μ is a function that takes elements from our σ-algebra as inputs and maps them to the (typically) extended real line, $\mathbb{R}^+$.

In this lecture we will cover where we might run into problems when attempting to define measures. Let's first take a look at a familiar example − the Reimann integral.

1.2. **Problem with the Riemann integral.**

We define the Riemann integral as

$$\int_a^b f(x)\, dx$$

And construct a partition of the domain, where $\Pi = \{\pi_i\}$ is the set of disjoint intervals within $[a, b]$. For each partition, π_i, we define a_i to be the infimum of $f(x)$ evaluated within that interval and b_i as the supremum. More formally stated, we have that

$$a_i = \inf_{x \in \pi_i} f(x)$$
$$b_i = \sup_{x \in \pi_i} f(x)$$

Moreover, we define the lower bound (LB) and upper bound (UB) given our set of intervals as

$$LB(\Pi) \equiv \sum_i a_i \cdot |\pi_i|$$
$$UB(\Pi) \equiv \sum_i b_i \cdot |\pi_i|$$

and we say that f is a Riemann integrable when $\sup_\Pi LB(\Pi) = \inf_\Pi UB(\Pi)$. The following two figures depicts the graphs for the LB vs the UB. The idea behind the Riemann integral is that as the number of partitions increases, the magnitude of our LB increases and the magnitude of our UB decreases until the two are equal.

FIGURE 1. Lower Bound

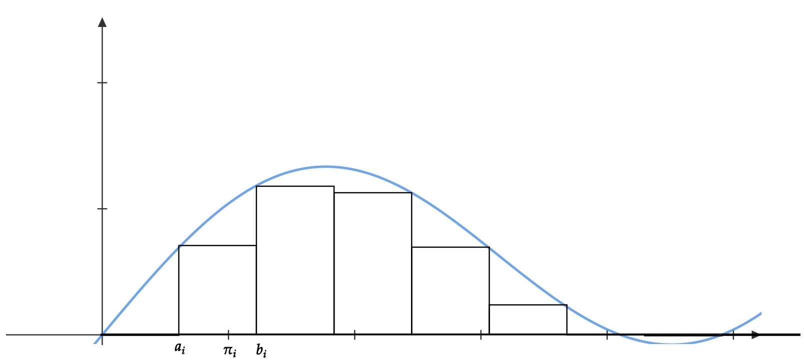
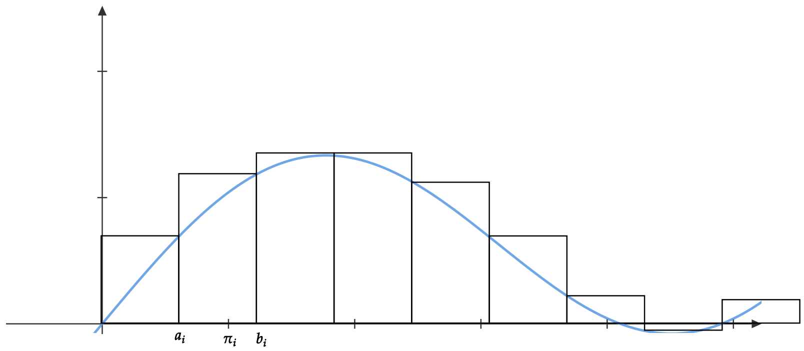


FIGURE 2. Upper Bound



Theorem 1. *Riemann Integrability.*

f is Riemann integrable if

(1) f is bounded

 (a) Set of discontinuities has Lebesgue measure equal to 0, i.e., there are a countable number of them.

There are several examples where the Riemann integral breaks. A classic example is the Dirichlet function.

$$f : [0,1] \to \{0,1\} : \begin{cases} 0 & x \in \mathbb{Q}^c \\ 1 & x \in \mathbb{Q} \end{cases}$$

In this example, f is discontinuous everywhere. For any partition you take, the $LB = 0$ and the $UP = 1$ $\therefore$ they will never equal.

The two main problems with the Riemann integral can be summarized as follows:

(1) Holds for a limited class of functions

(2) Basic limit operations don't hold

Example. Let's order the rationals $a_1, a_2, ...$ Now let's define the function

$$f_n(x) = \begin{cases} 1 & x \in \{a_1, \ldots, a_n\} \\ 0 & \text{else} \end{cases}$$

In this example, $f_n(x) \to f(x)$ point-wise where $f(x)$ is the Dirichlet function. But,

$$\lim_n \int f_n(x)\, dx \nrightarrow_? \int f(x)\, dx$$

does not exist.

1.3. **Defining a measure.** Formally, we defined a measure as a function from a family of sets to the real line (or extended real line). Let's consider the following example

$$\mu : 2^{\Omega} \to \mathbb{R}^{+}$$

where μ is a measure from the power set to the extended reals. For our measure to be well-defined, we want that at least the following three properties hold for $A, B \subset \Omega$.

(1) $\mu(\emptyset) = 0$. The measure of the empty set is equal to 0

(2) If $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$. If A and B are disjoint, then the measure of their union is equal to the sum of the measures of each.

(3) If $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. If A is contained in B then the measure of A is less than or equal to the measure of B. This is called monotonicity.

However, these definitions are not sufficient to construct a well-defined measure. Let's consider the following counterexample.

Example. Let $p \neq q \in \Omega$ and $A =\subseteq 2^{\Omega}$. Now, let's define our "measure"

$$\mu(A) = \begin{cases} 1 & \text{if } p, q \in A \\ 0 & \text{else} \end{cases}$$

Then, we see that $\mu(\{p\}) + \mu(\{q\}) = 0 + 0 \neq 1 = \mu(\{p, q\})$.

This result implies that μ cannot be any arbitrary function from the power-set.

Let's take an even closer look using a more common measure, the Lebesgue measure. We define the Lebesgue measure as

$$\mu_L([a, b]) = b - a$$

where $[a, b] \subseteq \mathbb{R}$. Let's consider a simple example where our loose definition of a measure fails using a nice set and a common measure

Example. Let $\Omega = [0, 1]$, $A_i = \{i\}$ for $i \in [0, 1]$. Clearly we have that $\mu_L(A_i) = 0$ and $\cup A_i = [0, 1]$. However, we see that property (2) of our definition fails

$$1 = \mu_L([0, 1]) = \mu_L\left(\cup_{i \in [0,1]} A_i\right) \neq \sum_{i \in I} \mu_L(A_i) = 0$$

The problem here being that $i \in I$ is uncountable.

To summarize, not all sets are measurable, and even if they are measurable we cannot deal with uncountable additivity. Therefore, throughout this course we will be dealing with countable additivity of measurable sets.

2. Rings, Algebras, and Sigma Algebras

In this lecture, we formally define what it means to be a measure and study properties for the families of sets which we will be using throughout the course.

2.1. **Defining measures.**

In the previous lecture, we gave a very loose definition of "measures", and we saw examples where this definition failed. Now, we will formally define what it means to be a measure.

Definition 5. A set function $\mu : \mathcal{A} \to [0, \infty]$ is a countably (also implying finitely) additive measure with respect to $\mathcal{A}$ if

(1) $\emptyset \in \mathcal{A}$ then $\mu(\emptyset) = 0$

(2) If $A_1, \ldots, A_n \in \mathcal{A}$ and $\cup_{i=1}^n A_i \in \mathcal{A}$ are pairwise disjoint, then $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. Finite additivity.

(3) If $\{A_i\}_{i \in I} \subseteq \mathcal{A}$ are disjoint and I is a countable index set and $\cup_{i \in I} A_i \in \mathcal{A}$, then $\mu(\cup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$.

You might be wondering about uncountably additivity and monotonicity. Recall from our final example in the previous lecture with the Lebesgue measure – our definition fails for when I is uncountable. And as for monotonicity, it is implied by finite additivity.

Proposition 1. *Finitely additive measures are monotone.*

Proof. If $A, B, B \backslash A \in \mathcal{A}$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$. We can say $\mu(B) = \mu(A \cup B \backslash A) = \mu(A) + (B \backslash A) \geq \mu(A) . \therefore \mu(B) \geq \mu(A)$ $\square$

Corollary 1. *Countably additive measures are also monotone*

Using similar techniques, we can also show that finitely additive measure are also finitely sub-additive

Proposition 2. *Finitely additive measures are finitely sub-additive.*

Proof. WTS $\mu(A \cup B) \leq \mu(A) + (B)$ provided that $A, B, A \cup B, B \backslash A \in \mathcal{A}$. We can say $\mu(A \cup B) = \mu(A \cup B \backslash A) = \mu(A) + \mu(B \backslash A) \leq \mu(A) + \mu(B) \therefore \mu(A \cup B) \leq \mu(A) + \mu(B)$. $\square$

A special type of measure which we will see throughout the course is a probability measure.

Definition 6. A measure is a probability measure if $\Omega \in \mathcal{A}$ and $\mu(\Omega) = 1$.

To understand when such measures exist, we need to examine the families of sets measures are built on.

2.2. **Families of sets.** All the definitions of measures that we've given require certain closure properties of the set $\mathcal{A}$. In this section, we will give more structure to $\mathcal{A}$, and our end goal will be to build a measure by taking advantage of the composition rules of $\mathcal{A}$.

Definition 7. $\mathcal{A} \subseteq 2^{\Omega}$ is a ring if

(1) $\emptyset \in \mathcal{A}$

(2) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

(3) $A, B \in \mathcal{A} \Rightarrow A \backslash B \in \mathcal{A}$

Definition 8. Notice that $A \cap B = A \backslash (A \backslash B) \in \mathcal{A}$. Therefore, by induction we can say that rings are closed under finite $\cup, \cap$, and relative complements.

Definition 9. $\mathcal{A}$ is an algebra/field if $\mathcal{A}$ is a ring such that $\Omega \in \mathcal{A}$.

Since $\mathcal{A}$ is a ring, we know it must also be closed under finite set operations

Definition 10. $\mathcal{A}$ is a σ-ring if $\mathcal{A}$ is a ring that is closed under countable unions.

Since $\mathcal{A}$ is a ring and we know rings are closed under relative complements, then we also know that σ-rings must also be closed under countable intersections. This is because $\cap_{n=1}^{\infty} A_n = A_1 \backslash (\cup_{n=1}^{\infty} A_1 \backslash A_n)$

Definition 11. $\mathcal{A}$ is a σ-algebra/σ-field if $\mathcal{A}$ is an algebra closed under countable unions

Definition 12. $\mathcal{A}$ is a semi-ring if

(1) $\emptyset \in \mathcal{A}$

(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

(3) $A \backslash B = \cup_{1 \leq j \leq n} C_j$, where $C_j \in \mathcal{A}$ are pairwise disjoint

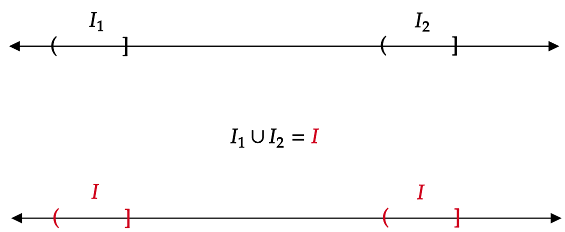
2.3. **Generating sets.** Now, let's discuss how we can generate families of sets from smaller families of sets. We'll first consider the set

$$\mathcal{I} = \{(a, b]\}$$

defined to be the set of all intervals in $\mathbb{R}$.

The first thing to notice is that $\mathcal{I}$ is not a ring. Consider the union of two disjoint $I_1, I_2 \in \mathcal{I}$. Their union is not an interval, rather it is a finite union of disjoint elements from $\mathcal{I}$.

FIGURE 3. $\mathcal{I}$ is not a ring



However, we do know that $\mathcal{I}$ is a semi-ring. In this example above, we can say that $\mathcal{I}$ is a semi-ring because it satisfies our definition of "relative complements".

Notice that we can easily define our Lebesgue "measure" on the semi-ring $\mathcal{I}$. We can just say that $\mu_{L_0}((a,b]) = b - a$. Our goal is to construct a ring from the semi-ring, extending our notion of the Lebesgue measure such that $\mu_{L_1}|_{\mathcal{I}} = \mu_{L_0}$.

Theorem 2. *For any semi-ring, $\mathcal{A}$, let $\mathcal{B}$ be the set of all finite disjoint unions of sets in $\mathcal{A}$. Then $\mathcal{B}$ is a ring.*

Proof. We know that $\mathcal{B}$ is a ring because it satisfies the definition $\square$

(1) $\emptyset \in \mathcal{B}$

(2) $A = \cup_{i=1}^{n} A_i$ disjoint, $B = \cup_{j=1}^{n} B_j$ disjoint where $A_i, B_i \in \mathcal{A}$, then by definition $A, B \in \mathcal{B}$. However, we can also say that $A \cap B = \cup_{i.j} A_i \cap B_j$ which is the union of finite, disjoint intervals so $A \cap B \in \mathcal{B}$.

(3) $B \backslash A \in \mathcal{B}$. We can say that $B \backslash A = \cup_{i,j} B_j \backslash A_i$ where $B_j \backslash A_i$ is disjoint so we're done.

(4) $A \cup B = A \cup (B \backslash A)$ is disjoint where $A \in \mathcal{B}, B \backslash A \in \mathcal{B}$ so their union is also in $\mathcal{B}$

Let's now define the set $\mathcal{J}$ to be the ring generated by the finite disjoint union of elements in $\mathcal{I}$. Now, let's extend μ_{L_0} on our semi-ring $\mathcal{I}$ to μ_{L_1} defined on our ring $\mathcal{J}$. If $J \in \mathcal{J}$ then we know that $J = \cup_{i=1}^{n} I_i$ where $I_i \in \mathcal{I}$ are disjoint. So let's define μ_{L_1} as

$$\mu_{L_1}(J) \triangleq \sum_{i=1}^{n} \mu_{L_0}(I_i)$$

Theorem 3. *μ_{L_1} on $\mathcal{J}$ is a finitely additive measure.*

Proof. We need to check that μ_{L_1}satisfies the conditions for what it means to be a finitely additive measure on a set. $\square$

(1) $\mu_{L_1}(\emptyset) = 0$

(2) $J_1, J_2 \in \mathcal{J}$ disjoint, then we have $J_1 = \cup_{i=1}^n I_{1i}$ and $J_2 = \cup_{i=1}^n I_{2j}$ where all I's are disjoint by definition. Then $\mu_{L_1}(J_1 \cup J_2) = \sum_i \mu_{L_0}(I_{1i}) + \sum_j \mu_{L_0}(I_{2j}) = \mu_{L_1}(J_1) + \mu_{L_1}(J_2)$.

Thus extending the μ_{L_0} on $\mathcal{I}$ to finitely additive measure μ_{L_1} on the ring $\mathcal{J}$.

Our goal now is to extend μ_{L_1} to be a countably additive measure on the σ-algebra.

3. Sigma Algebras and Outer Measures

In the previous lecture, we've shown how to extend the Lebesgue "measure" on the semi-ring of half open, half closed intervals to a finitely additive measure on the on the ring generated by the side of all finite disjoint unions of half open, half closed intervals. In this lecture, we will continue with this reasoning and try to extend our finitely additive Lebesgue measure on our ring to a countable additive measure on a σ algebra which contains our rings, and hence our semi-ring.

3.1. **Motivation.** Apart from showing that it's possible to extend our notion of a "measure" to larger families of sets, there's also a practice reason for doing – countably additive measures on σ-algebras allow us to take limits.

Proposition 3. *Countably additive measures are "continuous" on σ-algebras*

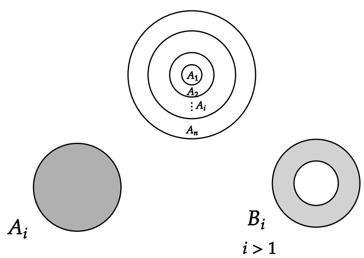
(1) $A_i \uparrow A$ ($A_1 \subseteq A_2 \subseteq$ ... and $\cup A_i = A$) $\Rightarrow \mu(A_i) \uparrow \mu(A)$

(2) $A_i \downarrow A$ and $\mu(A_1) < \infty$ ($A_1 \supseteq A_2 \supseteq$ ... and $\cap A_i = A$) $\Rightarrow \mu(A_i) \downarrow \mu(A)$

Let's look at the proof for (1).

Proof. Define $B_1 = A_1$ and $B_i = A_i \backslash A_{i-1}$ for $i > 1$ as depicted by the figure below.

Figure 4. Proposition 3 Part 1



Given how B_i is defined, we know they are disjoint and we know that $A_n = \cup_{i=1}^n B_i$, so $A = \cup_i B_i$. By monotonicity, we also know that $\mu(A_{n+1}) \leq \mu(A_n)$.

With what we know, we can say that $\mu(A) = \mu(\cup_i B_i)$ and by countable additivity we can say $\mu(\cup_i B_i) = \sum_i \mu(B_i) = \lim_n \sum_{i=1}^n \mu(B_i)$ and now by finite additivity we have $\lim_n \mu(A_n)$. ☐

3.2. **Generating larger families of sets.** In the previous lecture, we constructed a ring from a semi-ring. Now, we want to discuss generating *any* larger family of sets which contains some initial family $\mathcal{A} \subseteq 2^\Omega$.

For example, we say that ring $(\mathcal{A}) \triangleq$smallest ring that contains $\mathcal{A}$. Similarly for σ-ring($\mathcal{A}$), alg($\mathcal{A}$), or σ-alg($\mathcal{A}$) (which we will just denote by $\sigma(\mathcal{A})$). Slightly more formally, we say that the $*(\mathcal{A}) = \cap \mathcal{B}$ for all sets $\mathcal{B}$ such $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ has the desired properties of $\mathcal{A}$. Given this definition, let's go back to our previous example looking at the set of all half open, half closed intervals on the real line.

Proposition 4. *ring*$(\mathcal{I}) = \mathcal{J} = \{$*all finite disjoint union of elements in* $\mathcal{I}\}$.

Proof. Any ring $\mathcal{B}$ that contains $\mathcal{I}$ must also contain all finite disjoint unions. Hence, we know that $\mathcal{J} \subseteq \mathcal{B}$ for all such $\mathcal{B}$. Since we've showed that $\mathcal{J}$ is a ring, this confirms that $\mathcal{J}$ must also be the smallest ring. $\square$

Since our goal for the lecture is to construct a measure on a σ-algebra, naturally we may ask what is $\sigma(\mathcal{I})$? Consider the following definition.

Definition 13. Let $\mathcal{X}$ be a topological space (ex: $\mathbb{R}$). The Borel σ-algebra on $\mathcal{X}$ denoted by $\mathcal{B}(\mathcal{X})$ is the smallest σ-algebra that contains all open sets.

Proposition 5. *Let* $\mathcal{G} = \{$*all open sets in* $\mathbb{R}\}$. *Then* $\sigma(\mathcal{I}) = \sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$, *i.e., both* $\mathcal{I}$ *and* $\mathcal{G}$ *generate* $\mathcal{B}(\mathbb{R})$.

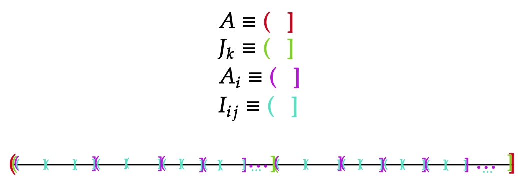
Proof. By definition, $\sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$. Now we need to show that $\forall I \in \mathcal{I}$, $I \in \sigma(\mathcal{G})$ and $\forall G \in \mathcal{G}, G \in \sigma(\mathcal{I})$. Since $\mathcal{I}$ is the set of all half-open, half-closed intervals,$I \in \mathcal{I}$ such that $I = (a, b]$, we can say that $(a, b] = \cap_{i=1}^{\infty} \left(a, b + \frac{1}{i}\right) \in \sigma(\mathcal{G})$ since σ algebras are closed under countable intersections. Thus, we know that $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{G})$. To show that $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{I})$, we need to use that fact that every nonempty open set in $\mathbb{R}$ is the disjoint union of a countable number of open intervals. Consider $G \in \sigma(\mathcal{G})$ such that $G = (a, b)$. Using that fact, we can say that $(a, b) = \cup_i \left(a, b - \frac{1}{i}\right] \in \sigma(\mathcal{I})$. Thus, we know that $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{I})$ and can conclude that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}) = \sigma(\mathcal{I})$. $\square$

Returning to our discussion of the Lebesgue measure, we've already shown that it is finitely additive on the ring($\mathcal{I}$). Now we will show that it is also countably additive on the ring($\mathcal{I}$).

Theorem 4. μ_{L_1} *is countably additive on the ring*$(\mathcal{I})$

Proof. Let $\{A_i\} \subseteq ring(\mathcal{I})$ be countable, and pairwise disjoint and let $A = \cup A_i \in ring(\mathcal{I})$. We want to show that $\mu_{L_1}(A) = \sum_i \mu_{L_1}(A_i)$. The proof for this can be split into steps. The first step is to show that $\mu_{L_1}(A) \geq \sum_i \mu_{L_1}(A_i)$. Then we will show that $\mu_{L_1}(A) \leq \sum_i \mu_{L_1}(A_i)$ which is more involved. For the first step, we know that $\cup_{i=1}^{n} A_i \subseteq A$, so by monotonicity we can say that $\mu_{L_1}(A) \geq \mu_{L_1}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu_{L_1}(A_i)$ by finite additivity on the ring($\mathcal{I}$). Now just let $n \uparrow \infty$ and we're done. For step 2, let's consider the following: let $A = \cup_{k=1}^{K} J_k$ be finite disjoint unions of $J_k \in \mathcal{I}$ and let $\cup_i A_i = \cup_i \cup_{j=1}^{n_i} I_{ij}$ be countable and disjoint since all A_i are disjoint and $I_{ij} \in \mathcal{I}$. The figure below represents such a construction.

Figure 5. Therem 4 Step 2



By finite additivity, we know that $\mu_{L_1}(A) = \sum_{k=1}^{K} \mu_{L_1}(J_k)$. And again by finite additivity we can say that $\sum_i \mu_{L_1}(A_i) = \sum_i \sum_{j=1}^{n_i} \mu_{L_1}(I_{ij})$. Since our measures are ≥ 0, we can say $\sum_i \sum_{j=1}^{n_i} \mu_{L_1}(I_{ij}) = \sum_{k=1}^{K} \sum_{i,j} \mu_L(I_{ij})$ such that $I_{ij} \subseteq J_k$. Recall that our goal is to show that $\mu_{L_1}(A) \leq \sum_i \mu_{L_1}(A_i) \Rightarrow \sum_{k=1}^{K} \mu_{L_1}(J_k) \leq \sum_{k=1}^{K} \sum_{i,j} \mu_{L_1}(I_{ij})$. If we can show that for some arbitrary J_k that $\mu_{L_1}(J_k) \leq \sum_{i,j} \mu_{L_1}(I_{ij})$ then we will have completed our proof. Let's choose some interval $J = (a, b]$, to restate our goal we want that $\mu_{L_1}((a,b]) \leq \sum_l \mu_{L_1}((a_l, b_l])$ where the intervals $\{(a_l, b_l]\}$ represent a countable, disjoint cover for $(a, b]$. Our goal is to extract a finite cover for the set $(a, b]$. We can say the following,

$$(a+\epsilon, b] \subseteq [a+\epsilon, b] \subseteq (a, b] = \cup_l (a_l, b_l]$$
$$(a_l, b_l] \subseteq \left(a_l, b_l + \frac{\epsilon}{2^l}\right) \subseteq \left(a_l, b_l + \frac{\epsilon}{2^l}\right]$$

which implies that $[a+\epsilon, b] \subseteq \cup_l \left(a_l + b_l + \frac{\epsilon}{2^l}\right)$ and by compactness we know that there must exist a finite cover such that $[a+\epsilon, b] \subseteq \cup_{l\in L} \left(a_l, b_l + \frac{\epsilon}{2^l}\right)$ where L is a finite index set. By the steps above, we now have

$$\begin{aligned}
(a+\epsilon, b) \subseteq \cup_{l\in L} \left(a_l, b_l + \frac{\epsilon}{2^l}\right) &\Rightarrow \\
b - (a+\epsilon) &\leq \sum_{l\in L} \left(b_l + \frac{\epsilon}{2^l} - a_l\right) \text{(monotonicity and finite subadd.)} \\
b - a &\leq \sum_{l\in L} (b_l - a_l) + \sum_{l\in L} \frac{\epsilon}{2^l} + \epsilon \\
b - a &\leq \sum_{l} (b_l - a_l) + 2\epsilon \text{(take } l \text{ to be countable)} \\
\mu_{L_1}((a,b]) &\leq \sum_{l} \mu_{L_1}((a_l, b_l]) \text{ (take } \epsilon \downarrow 0)
\end{aligned}$$

Repeating this process for each J_k, we will have that $\mu_L(A) \leq \sum \mu_l(A_i)$ which implies μ_L is countably additive on ring(I) □

Our new goal, is to extend our notion of μ_{L_1} to the $\sigma(I)$. To do so, we need to develop the idea of an **outer-measure**.

Definition 14. Let $\mu^* : 2^\Omega \to \mathbb{R}^+ \cup \{+\infty\}$ be an outer-measure if

(1) $\mu^*(\emptyset) = 0$

(2) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(3) If $A \subseteq \cup_i A_i$ is countably and not necessarily disjoint, then $\mu^*(A) \leq \sum \mu^*(A_i)$, i.e. is countable sub-additive

Theorem 5. *Carathéodory Extension Theorem. Given a countably additive measure, μ, on the ring $\mathcal{A}$. Let*

$$\mathcal{C} = \{A \subseteq \Omega \ s.t \ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subseteq \Omega\}$$

call element $A \in \mathcal{C}$ the measurable set.

(1) $\mathcal{A} \subseteq \mathcal{C}$ $(\Rightarrow \sigma(\mathcal{A}) \subseteq \mathcal{C})$

(2) $\mathcal{C}$ is a σ-algebra

(3) μ^* is a countably additive measure on $\mathcal{C}$ (and hence on $\sigma(\mathcal{A})$)

(4) μ^* respects μ on $\mathcal{A}$ ($A \in \mathcal{A} \Rightarrow \mu^*(A) = \mu(A)$).

Now, let's introduce how to construct the outer-measure. Given a countably additive measure μ on ring $\mathcal{A}$, then $\forall E \subseteq \Omega$ let

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \text{ s.t. } A_i \text{ are countable cover of } E, A_i \in \mathcal{A}, \cup A_i \supseteq E \right\}$$

If no such cover exists, then we $\mu^*(E) = \infty$

4. Caratheodory's Extension Theorem

In this lecture, we continue our discussion of outer-measures and Carathéodory's extension theorem. Carathéodory's extension theorem is so powerful because it tells us that if we have a measure on a ring then we know there exists a unique extension of that measure to a sigma algebra, so long as certain conditions are met.

4.1. **Construction of an outer-measure.** Let's recall how we can construct an outer-measure given a countably additive measure μ on a ring $\mathcal{A}$. For all $E \subseteq \Omega$, we have that

$$\mu^*(E) = \inf\left\{\sum_i \mu(A_i) \text{ where } A_i \in \mathcal{A} \text{ countable }, E \subseteq \cup A_i \text{ cover }\right\}$$

and if no such cover exits then $\mu^*(E) = \infty$

Theorem 6. *The construction defined above is, in fact, an outer-measure.*

(1) $\mu^*(\emptyset) = 0$

(2) $E_1 \subseteq E_2 \subseteq \Omega \Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$

(3) $E_{i,} E \subseteq \Omega$ s.t. $E \subseteq \cup E_i$ (countable) $\Rightarrow \mu^*(E) \leq \sum \mu^*(E_i)$

(4) If $A \in \mathcal{A}$ then $\mu^*(A) = \mu(A)$, i.e., μ^* respects μ on $\mathcal{A}$.

Proof. Let's prove each statements individually. □

(1) Let each $A_i = \emptyset$, then $\{A_i\}$ is a cover of $\emptyset$ and $\sum_i \mu(\emptyset) = 0$.

(2) Any cover of E_2 must also cover E_1.

(3) If $\sum \mu^*(E_i) = \infty$ then there is nothing to prove. If $\sum \mu^*(E_i) \leq \infty \Rightarrow \mu^*(E_i) < \infty, \forall i$, however, then we require more. Fix $\epsilon > 0$, choose $A_{ij} \in \mathcal{A}$ s.t. $E_i \subseteq \bigcup_j A_{ij}$, then

$$\mu^*(E_i) \leq \sum_j \mu(A_{ij}) < \mu^*(E_i) + \frac{\epsilon}{2^i}$$

which follows by the definition of the infimum. Since $E \subseteq \bigcup_i E_i \subseteq \bigcup_i \bigcup_j A_{ij}$, then we know that

$$\mu^*(E) \leq \sum_{ij} \mu(A_{ij}) \leq \sum_i \left(\mu^*(E_i) + \frac{\epsilon}{2^i}\right) = \sum_i \mu^*(E_i) + \epsilon$$

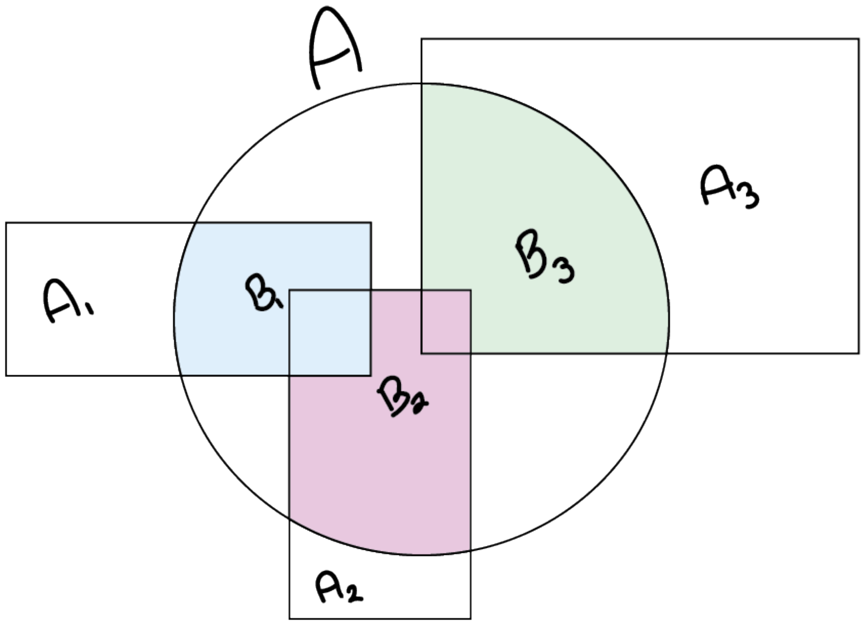
now let $\epsilon \downarrow 0$, and we have

$$\mu^*(E) \leq \sum_i \mu^*(E_i)$$

(4) Let $A \in \mathcal{A}$ show $\mu^*(A) = \mu(A)$. First let's show that $\mu^*(A) \leq \mu(A)$. Let $A_1 = A$, and $A_i = \emptyset$ for $i = 2, 3, ...$ so $\{A_i\}$ is a countable cover of A. This implies that $\mu^*(A) \leq \sum_i \mu(A_i) = \mu(A_1) = \mu(A) \Rightarrow \mu^*(A) \leq \mu(A)$. For the opposite direction, $\mu^*(A) \geq \mu(A)$,

we want to show that all covers are $\geq \mu(A)$. Let $\{A_i\} \subseteq \mathcal{A}$ be a countable cover of $\mathcal{A}$, i.e., $A \subseteq \bigcup_i A_i$. Now let $B_i = A \bigcap \left(A_i \backslash \bigcup_{j<i} A_j \right)$ and consider the following diagram below,

FIGURE 6. A and B_i Illustration



In the figure, we did not complete the cover $\{A_i\}$ of A, but we get a general sense for how the B_i are supposed to be interpreted. Once we include the entire cover of A however, notice that we can say that B_i are disjoint, $B_i \in \mathcal{A}$, $B_i \subseteq A_i$ and $A = \bigcup B_i$. This implies that $\mu(A) = \sum \mu(B_i)$ by μ countable additivity on $\mathcal{A}$ and since $A \subseteq \bigcup A_i$, we know by monotonicity that $\mu(A) = \sum \mu(B_i) \leq \sum_i \mu(A_i)$ which generalizes to be true for all covers $\{A_i\}$ thus accomplishing our goal in showing that $\mu(A) \leq \mu^*(A)$ and concluding our proof that $\mu(A) = \mu^*(A)$.

Our goal for this section is to find a σ-algebra, $\mathcal{F} \subseteq 2^{\Omega}$ s.t. $\mathcal{A} \subseteq \mathcal{F}$ and μ^*, the outer measure defined in 2^{Ω}, is a countably additive measure on $\mathcal{F}$.

4.2. **Caratheodory's Extension Theorem.** Given a countably additive measure μ defined on the ring $\mathcal{A}$. Let μ^* be the outer measure that respects μ on $\mathcal{A}$. Now let

$$\mathcal{C} = \{A \subseteq \Omega \text{ s.t } \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subseteq \Omega\}$$

and call sets in $\mathcal{C}$ the **measurable sets.** Then,

(1) $\mathcal{A} \subseteq \mathcal{C}$

(2) $\mathcal{C}$ is a σ-alg

(3) μ^* is a countably additive measure on $\mathcal{C}$ and hence on $\mathcal{A}$

Let's make a few observations regarding the statement. The first is that (1) and (2) imply that $\mathcal{A} \subseteq \sigma(\mathcal{C}) = \mathcal{C} \Rightarrow \sigma(\mathcal{A}) \subseteq \mathcal{C}$. This means that $\mathcal{C}$ is a σ-algebra that contains $\sigma(\mathcal{A})$, but it's not necessarily the smallest. We also know that by countable sub-additivity of μ^*, we must always have that when $E \subseteq \cup E_i \Rightarrow \mu^*(E) \leq \sum \mu^*(E_i)$. In this example, we can just let $E_1 = E \cap A$ and $E_2 = E \cap A^c$ and we have that $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Therefore, the difficulty arises in relationship to the Carathéodory Extension Theorem when showing equality.

Let's consider the following example. Let $A \bigcap B = \emptyset$ and let $A \in \mathcal{C}$, so we know it's a measurable set. Then we can say the following

$$\begin{aligned} \mu^*(A \cup B) &= \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) \\ &= \mu^*(A) + \mu^*(B) \end{aligned}$$

i.e., they are equal in subadditivity.

Now, we will attempt to cover the intuition for splitting. Recall for the Riemann integral, the goal was to take the limit such that we the upper bound equal to the lower bound. Similarly, we consider the outer measure, $\mu^*(A)$ and the "inner" measure which we'll define by

$$\mu_*(A) = \mu^*(\Omega) - \mu^*(A^c)$$

If these two objects are equal, then we'll have $\mu^*(\Omega) = \mu^*(A) + \mu^*(A^c)$ as desired. Since, there's nothing inherently special about Ω, we can split it up with respect to any $E \subseteq \Omega$ such that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Recall that by the very definition of the outer-measure, we are given subadditivity for free ($\leq$), but now we need to prove ($\geq$). Let assume that $\mu^*(E) < \infty$ or else the proof is trivial.

Proof. Proof of Carthéodory's Extension Theorem. We first need to show that $\mathcal{A} \subseteq \mathcal{C}$, i.e., if $A \in \mathcal{A}$ then $A \in \mathcal{C}$.

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subseteq \Omega$$

We already know that ($\leq$) is given by subadditivity so we need to go about showing ($\geq$). Let's fix $E \subseteq \Omega$ such that $\mu^*(E) < \infty$ and fix $\varepsilon > 0$. Take $\{A_i\} \subseteq \mathcal{A}$ be a countable cover of E such that (1) $E \subseteq \cup_i A_i$ and (2)$\mu^*(E) \leq \sum \mu(A_i) < \mu^*(E) + \varepsilon$. We just need to show that the cover can be used to cover $E \cap A$ and $E \cap A^c$. We know that $\{A_i\}$ is a countable cover for E, therefore we know that $E \cap A \subseteq \cup_i (A_i \cap A)$ and $E \cap A^c \subseteq \cup_i (A_i \cap A^c)$. We can also say that $A_i \cap A$ and $A_i \cap A^c$ belong to the ring $\mathcal{A}$ where recall $\mu(A)$ is a defined as a countably additive measure. Now we can take all of our pieces and say that

$$\begin{aligned} \mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_i [\mu(A_i \cap A) + \mu(A_i \cap A^c)] \\ &= \sum_i \mu(A_i) \\ &\leq \mu^*(E) + \varepsilon \text{ (by our definition)} \end{aligned}$$

Now just let $\varepsilon \downarrow 0$ and we finally have that $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$ thus completing our proof for $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ and show that $\forall A \in \mathcal{A} \Rightarrow A \in \mathcal{C}$. Now we must show that $\mathcal{C}$ is a σ-algebra. We will now check that each condition is satisfied. (1) $\emptyset \in \mathcal{C}$. Let $A = \emptyset$, then $\mu^*(E) = \mu^*(\emptyset) + \mu^*(E \cap \Omega) = \mu^*(E)$. (2) If $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$. This is true by the symmetric definition of the measurable sets. (3) If $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$. We know that $A \in \mathcal{C}$, so we have that

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*(E \cap A^c) \text{ (since we know } B \in \mathcal{C}) \\
&= \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B)^c \cap A) + \mu^*(E \cap (A \cap B)^c \cap A^c) \\
&= \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \cap B)^c)
\end{aligned}$$

thus showing that $A \cap B \in \mathcal{C}$. And since $A \cap B = (A^c \cap B^c)^c$, then we know that $\mathcal{C}$ is closed under complements so it must be an algebra. To show that it's a σ-algebra, we need to verify that $\mathcal{C}$ is also closed under countable unions. Let's consider the set $\{A_i\} \subseteq \mathcal{C}$ and let $B = \cup A_i$ such that $B \in \mathcal{C}$. Define $B_n \triangleq \cup_{i=1}^n A_i \in \mathcal{C}$ by finite $\cup$. WLOG, assume that A_i are disjoint, thus B_n is a disjoint union of $A_1, ..., A_n$. We now have that

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n^c) + \mu^*(E \cap B_n) \\
&= \mu^*(E \cap B_n^c) + \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n + A_n^c) \\
&= \mu^*(E \cap B_n^c) + \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})
\end{aligned}$$

by recursion, this implies that

$$\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\
&= \mu^*(E \cap B_n^c) + \sum_{i=1}^n \mu^*(E \cap A_i) \\
&\geq \mu^*(E \cap B^c) + \sum_{i=1}^n \mu^*(E \cap A_n) \\
\mu^*(E) &\geq \mu^*(E \cap B^c) + \sum_i \mu^*(E \cap A_n) \\
&\geq \mu^*(E \cap B^c) + \mu^*(E \cap B)
\end{aligned}$$

and we have $\leq$ by sub-additivity so we know that countable unions are in $\mathcal{C}$.

Now, we want to show that μ^* is countably additive on the σ-algebra $\mathcal{C}$. Let $\{A_i\} \subseteq \mathcal{C}$ be disjoint, countable. Let $B = \cup A_i \in \mathcal{C}$. We have that $\mu^*(B) \leq \sum \mu^*(A)$ by countable sub-additivity but now we want that $\mu^*(B) \geq \sum \mu^*(A)$. We know from the previous proof that $\mu^*(E) \geq \mu^*(E \cap B^c) + \sum_i \mu^*(E \cap A_i)$. Now let $E = B$ and we have

$$\begin{aligned}
\mu^*(B) &\geq 0 + \sum_i \mu^*(B \cap A_i) \\
&= \sum_i \mu^*(A_i)
\end{aligned}$$

thus completing our proof. □

We have finally constructed a countably additive measure on a σ-algebra based on a countably additive measure on a ring.

4.3. **Existence and Uniqueness.** Now, we want to ask under what conditions the measures we've constructed on our σ-algebra exist and unique. Let μ be a countably additive measure on the algebra $\mathcal{A}$. Let λ be a countably additive measure on the $\sigma(\mathcal{A})$ that respects μ on $\mathcal{A}$. We actually already know that λ exists by the Cartheorody Theorem, but is it unique? The answer is no!

Consider the following counter example. Let $\mathcal{A} = \text{alg}(\mathcal{I})$ and let $\mu(A) = +\infty$ whenever $A \neq \emptyset$. Now, let's define $\mathcal{B} = \sigma(\mathcal{A})$. For $B \in \mathcal{B}$ let $\lambda_1(B) = +\infty$ whenever $B \neq \emptyset$. Note that λ_1 respects μ on $\mathcal{A}$. But now also let $\lambda_2(B) =$ counting measure. Note, $\lambda_2(B)$ also respects μ on $\mathcal{A}$ because counting the number of elements in an interval is uncountably infinite. However, notice the following difference when considering singletons

$$\lambda_1(\{x\}) = +\infty$$
$$\lambda_2(\{x\}) = 1$$

Why does this happen? It's because the algebra of intervals actually does not contain singletons. In the section, we will address how to fix this problem.

5. Completion of a Measure

This section completes our construction of countably-additive measure on the α-algebra. We first discuss how to ensure uniqueness of a measure and then broaden our application of measures to measurable functions.

5.1. **Existence and Uniqueness.** Recall that Cartheodory's extension theorem allowed us to go from a countably additive measure on the ring $\mathcal{A}$, to a countably sub-additive outer-measure on the power set 2^Ω, and finally the countably additive outer-measure $\mu^*|_\mathcal{C}$, where $\mathcal{C}$ is the sigma algebra of measurable sets which respects our measure μ on our ring.

At the end of the previous lecture, we showed a counter example for uniqueness using two different "measures". In this lecture, we'll go over how to fix this to guarantee uniqueness.

Definition 15. Let μ be a countably-additive measure on the algebra $\mathcal{A}$. We say μ is σ-finite if $\exists \{A_n\} \subseteq \mathcal{A}$ s.t.

(1) $\mu(A_n) < \infty, \forall n$

(2) $\Omega = \cup A_n$

For example: μ_{Leb} on $\mathbb{R}$ is σ-finite (let $A_n = (n, n+1]$ but μ_{count} on $\mathbb{R}$ is not since we cannot express $\mathbb{R}$ as a countable unite of finite sets.

Theorem 7. *Cartheodory-Hahn: Let μ be countably additive on the ring $\mathcal{A}$.*

(1) $\mu^*|_\mathcal{C}$ is a countably additive measure on a σ-algebra

(2) If μ is σ-finite then $\mu^*|$ is a unique measure on $\mathcal{C}$that respects μ on $\mathcal{A}$

5.2. **Completion.** Now that we've shown that if our measure μ defined on ring, $\mathcal{A}$,is σ-finite, then we know that there must exist a unique measure μ^* defined on a σ-algebra, $\mathcal{C}$, that respects μ on $\mathcal{A}$, and we can move to our next topic which is completion.

Definition 16. A measure space $(\Omega, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all subsets of sets of measure 0.

Proposition 6. *If $A \subseteq \Omega$ s.t. $\mu^*(A) = 0 \Rightarrow A \in \mathcal{C}$.*

Proof. Let's assume that there exists some $A \subseteq \Omega$ such that $\mu^*(A) = 0$. We want to show that A is in our σ-algebra of measurable sets, $\mathcal{C}$. For this to be true, then $\forall E \subseteq \Omega$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

(1) $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E$ by subadditivity of μ^*

(2) For the other direction

$$\begin{aligned} \mu^*(E) &\geq \mu^*(E \cap A^c) \text{ by monotonicity} \\ &= \mu^*(E \cap A^c) + \mu^*(A) \text{ by hypothesis} \\ &\geq \mu^*(E \cap A^c) + \mu^*(E \cap A) \text{ by monotonicity} \end{aligned}$$

Thus, we can conclude that $A \in \mathcal{C}$. □

Theorem 8. *$\mathcal{F}_{Leb}$ is the completion of $\mathcal{B}(\mathbb{R})$ and the measurable space is given by $(\mathbb{R}, \mathcal{F}_{Leb}, \mu_{Leb})$, and $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}_{Leb}$*

We say that the completion of a σ-algebra, $\mathcal{A}$, is the smallest complete σ-algebra that contains $\mathcal{A}$.

5.3. **Outer and inner approximation of Lebesgue measurable sets.** When talking the Lebesgue measure, most people mean with respect to $\mathcal{B}(\mathbb{R})$. The difference comes from the inclusion of all the measure zero sets $(\mathcal{F}_{Leb})$.

Theorem 9. *Let $A \subseteq \mathbb{R}$. Then A is Lebesgue measurable iff any of the following hold:*

(1) $\forall \varepsilon > 0$, $\exists$ open set $O \supseteq A$ s.t. $\mu^*(O \backslash A) < \varepsilon$ (outer method)

(2) $\exists G_\delta$ set $G \supseteq A$ s.t. $\mu^*(G \backslash A) = 0$ where G_δ is a countable intersection of open sets (outer method)

(3) $\forall \varepsilon > 0$, $\exists$ closed set $F \subseteq A$ s.t. $u^*(A \backslash F) < \varepsilon$ (inner method)

(4) $\exists F_\delta$ set $F \subseteq A$ s.t. $\mu^*(A \backslash F) = 0$ where F_δ is a countable union of closed sets.(inner method)

The main takeaway from Theorem 9 is that one can approximate any Lebesgue measurable set by $\mathcal{B}(\mathbb{R})$ arbitrarily closely with respect to the outer measure, μ^*.

Proof. First, let's show that if A is Lebesgue measurable then any of the 4 instances above hold. Lets consider the case where $\mu^*(A) < \infty$, then $\forall \varepsilon > 0$, $\exists$ a countable cover,$\{I_k\} \subseteq \mathcal{I}$, of A such that $\mu^*(A) \le \sum_k \mu_L(I_k) < \mu^*(A) + \varepsilon$. If $I_k = (a_k, b_k]$ and let $O_k = \left(a_k, b_k + \frac{\varepsilon}{2^k}\right)$ be open. This implies that $A \subseteq \cup I_k \subseteq \cup O_k \equiv O$. We can say,

$$\begin{aligned}
\mu^*(O) &\le \sum \mu^*(O_k) \text{ (by sub-additivity of the outer measure)} \\
&\le \sum \mu^*\left(\left(a_k, b_k + \frac{\varepsilon}{2^k}\right]\right) \text{ (by monotonicity)} \\
&= \sum \mu_L(I_k) + \frac{\varepsilon}{2^k} \\
&\le [\mu^*(A) + \varepsilon] + \varepsilon
\end{aligned}$$

By assumption, $\mu^*(A) < \infty$, so we know that $\mu^*(O) - \mu^*(A) \le 2\varepsilon$. We also assume that A is Lebesgue measurable, therefore by the Cartheodory theroem we know that

$$\begin{aligned}
\mu^*(O) &= \mu^*(O \cap A) + \mu^*(O \cap A^c) \\
&= \mu^*(A) + \mu^*(O \backslash A) \\
&\Rightarrow \\
\mu^*(O \backslash A) &= \mu^*(O) - \mu^*(A) \le 2\varepsilon
\end{aligned}$$

Thus completing our proof for the case when $\mu^*(A) < \infty(1)$. Now let's consider when $\mu^*(A) = \infty$. Let $A_i = A \cap (i, i+1]$ then A_i is measurable $\mu^*(A_i) < \infty$. Then $\forall A_i$ $\exists O_i$ open s.t. $A_i \subseteq O_i$ and $\mu^*(O_i \backslash A_i) \le \frac{\varepsilon}{2^i}$. Let $O = \cup O_i$. Then $O \backslash A = (\cup O_i) \backslash A \subseteq \cup_i (O_i \backslash A_i)$.This implies that $\mu^*(O \backslash A) \le \sum \mu^*(O_i \backslash A_i) \le \sum \frac{\varepsilon}{2^i} = 0$, thus concluding our proof for (1).

Now, let's show that (1) $\Rightarrow$ (2) $\Rightarrow$ A is Lebesgue measurable. For every k choose $O_k \supseteq A$ such that $\mu^*(O_k \backslash A) \le \frac{1}{k}$. Let $G = \cap_k O_k \in G_\delta$. We now have that $G \backslash A \subseteq O_k \backslash A, \forall k$. Hence, we can say that $\mu^*(G \backslash A) \le \mu^*(O_k \backslash A) \le \frac{1}{k}$ and let $k \uparrow \infty$. Thus, know that there exists some $G \in G_\delta$ such that $\mu^*(G \backslash A) = 0$. This implies that $G \backslash A$ is measurable because we know that the lebesgue measure is complete. Hence, we can say that $A = G \cap (G \backslash A)^c$ and since G and $G \backslash A$ is measurable, then A must be measurable. □